Multiple Zeros of Nonlinear Systems

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September 6, 2009

Abstract

As an attempt to bridge between numerical analysis and algebraic geometry, this paper formulates the multiplicity for the general nonlinear system at an isolated zero, presents an algorithm for computing the multiplicity strucuture, proposes a depth-deflation method for accurate computation of multiple zeros, and introduces the basic algebraic theory of the multiplicity. Furthermore, this paper elaborates and proves some fundamental theorems of the multiplicity, including local finiteness, consistency, perturbation invarance, and depth-deflatability. The proposed algorithms can accurately compute the multiplicity and the multiple zeros using floating point arithmetic even if the nonlinear system is perturbed.

1 Introduction

Solving a system of nonlinear equations in the form

$$\mathbf{f}(\mathbf{x}) \;=\; \mathbf{0} \quad \text{or} \quad \begin{cases} f_1(x_1,\ldots,x_s) &= 0 \\ \vdots & \vdots \\ f_t(x_1,\ldots,x_s) &= 0 \end{cases} \tag{1}$$

with $\mathbf{f}=[f_1,\ldots,f_t]^{\mathsf{H}}$ and $\mathbf{x}=(x_1,\ldots,x_s)$ is one of the most fundamental problems in scientific computing, and one of the main topics in most numerical analysis textbooks. In the literature, however, an important question as well as its answer seem to be absent over the years: What is the multiplicity of an isolated zero to the system and how to identify it.

For a single equation $f(x)=0$, it is well known that the multiplicity of a zero x_* is m if

$$f(x_*) \;=\; f'(x_*) \;=\; \cdots \;=\; f^{m\text{-}1}(x_*) \;=\; 0 \quad \text{and} \quad f^{(m)}(x_*) \;\neq\; 0. \tag{2}$$

The multiplicity of a polynomial system at a zero has gone through rigorous formulations since Newton's era [6, pp. 127-129] as one of the oldest subjects of algebraic geometry. Nonetheless, the standard multiplicity formulation and identification via Gröbner bases for polynomial systems are somewhat limited to symbolic computation, and largely unknown to numerical analysts.

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As an attempt to bridge between algebraic geometry and numerical analysis, we propose a rigorous formulation for the multiplicity structure of a general nonlinear system at a zero. This multiplicity structure includes, rather than just a single integer for the multiplicity, several structural invariances that are essential in providing characteristics of the system and the computation of the zero. For instance, at the zero $\mathbf{x}_* = (0,0)$ of the nonlinear system

$$\begin{cases} \sin x_1 \cos x_1 - x_1 &= 0 \\ \sin x_2 \sin^2 x_1 + x_2^4 &= 0 \end{cases} \tag{3}$$

we shall have:

- The *multiplicity* $m = 12$.
- Under a small perturbation to the system (3), there is a cluster of exactly 12 zeros (counting multiplicities) in a neighborhood of $\mathbf{x}_* = (0,0)$.
- The *Hilbert function* $\{1,2,3,2,2,1,1,0,0,\cdots\}$ forms a partition of the multiplicity 12.
- There exist 12 linearly independent differential operators ∂_{00}, ∂_{10}, $\ldots$, $\partial_{05} - \partial_{22}$, $\partial_{06} - \partial_{23}$, grouped by the differential orders and counted by the Hilbert function as shown in Figure 1 below. They induce 12 differential functionals that span the *dual space* associated with the system (3). Here, the differential operator

$$\partial_{j_1 \cdots j_s} \equiv \partial_{x_1^{j_1} \cdots x_s^{j_s}} \equiv \frac{1}{j_1! \cdots j_s!} \frac{\partial^{j_1+\cdots+j_s}}{\partial x_1^{j_1} \cdots \partial x_s^{j_s}} \tag{4}$$

  naturally induces a linear functional

$$\partial_{j_1 \cdots j_s}[\mathbf{x}_*] \; : \; f \longrightarrow (\partial_{j_1 \cdots j_s} f)(\mathbf{x}_*) \tag{5}$$

  on s-variate functions f with the existence of the indicated partial derivative at the zero $\mathbf{x}_*$. These functionals satisfy a *closedness condition* and vanish on the two functions in (3) at the zero $(0,0)$.
- The *breadth*, or the nullity of the Jacobian at $\mathbf{x}_*$, is 2.
- The *depth*, which is the highest differential order of the functionals at $\mathbf{x}_*$, is 6.

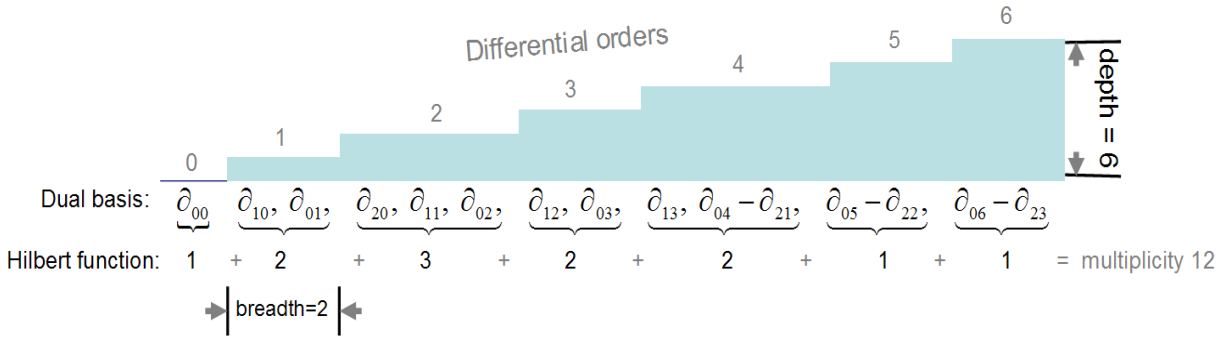

Figure 1: Illustration of the multiplicity structure, including dual basis, Hilbert function, breadth and depth of the system (3) at the zero $(0,0)$

Such a multiplicity structure at an isolated zero of a general nonlinear system will be introduced in §2. The theoretical foundation of those structures will be established in §4. In particular, we prove the so-defined *multiplicity* agrees with the *intersection multiplicity* of polynomial systems

in algebraic geometry. It is finite if and only if the zero is isolated, and more importantly, this finiteness ensures termination of the multiplicity identification algorithm NonlinearSystemMultiplicity given in §2.3, and it also provides a mechanism for determining whether a zero is isolated [2]. Furthermore, the multiplicity structure of the given nonlinear system can be computed by constructing the Macaulay matrices [14] together with the numerical rank revealing [13].

It is well documented that multiple zeros are difficult to compute accurately even for a single equation. There is a perceived barrier of "attainable accuracy": The number of correct digits attainable for a multiple zero is bounded by the number of digits in the hardware precision divided by the multiplicity. For instance, only three correct digits can be expected in computing a five-fold zero using the double precision (16 digits) floating point arithmetic. Such a barrier has been overcome for univariate polynomial equations [27]. Based on the multiplicity theory established in this article, we shall derive a *depth-deflation* algorithm in §3 for computing multiple zeros of general nonlinear systems, which can accurately compute the multiple zeros without extending the arithmetic precision even when the nonlinear system is perturbed. The *depth* defined in the multiplicity structure actually bounds the number of deflation steps. A related *multiplicity deflation* method is used in [11] for the main goal of speeding up Newton's iteration.

As mentioned above, the study of the multiplicity for a polynomial system at an isolated zero can be traced back to Newton's time [6, pp. 127-129]. Besides polynomial systems, multiple zeros of a nonlinear system occur frequently in scientific computing. For instance, when the system depends on certain parameters, a multiple zero emerges when the parameters reach a bifurcation point [3, §1.1]. Accurate computation of the multiple zero and reliable identification of the multiplicity structure may have a profound ramification in scientific computing.

This paper furnishes the theoretical details of the preliminary results on polynomial systems announced in the form of an abstract [5], and in addition, the scope of this work has been substantially expanded to general nonlinear systems.

2 Formulation and computation of the multiplicity structure

2.1 The notion and fundamental theorems of the multiplicity

The general nonlinear system (1) is represented by either the mapping $\mathbf{f} : \mathbb{C}^s \longrightarrow \mathbb{C}^t$ or the set $F = \{f_1, \ldots, f_t\}$ of functions in variables $x_1, \ldots, x_s$. We assume functions $f : \mathbb{C}^s \longrightarrow \mathbb{C}$ in this paper have all the relevant partial derivatives arising in the elaboration. The multiplicity which we shall formulate in this section will extend both the multiplicity (2) of a single equation and the Macaulay-Gröbner duality formulation of multiplicity for polynomial systems.

Denote $\mathbb{N} = \{0, \pm 1, \pm 2, \ldots\}$. For an integer array $\mathbf{j} = (j_1, \ldots, j_s) \in \mathbb{N}^s$, write $\mathbf{j} \geq 0$ if $j_i \geq 0$ for all $i \in \{1, \ldots, s\}$. For every $\mathbf{j} = (j_1, \cdots, j_s) \in \mathbb{N}^s$ with $\mathbf{j} \geq 0$, denote

$$\mathbf{x}^{\mathbf{j}} \;=\; x_1^{j_1} \cdots x_s^{j_s} \qquad \text{and} \qquad (\mathbf{x} - \mathbf{y})^{\mathbf{j}} \;=\; (x_1 - y_1)^{j_1} \cdots (x_s - y_s)^{j_s},$$

and differential functional monomial $\partial_{\mathbf{j}}[\hat{\mathbf{x}}]$ at $\hat{\mathbf{x}} \in \mathbb{C}^s$ as in (5), with order $|\mathbf{j}| = j_1 + \cdots + j_s$. For simplicity, we adopt the convention

$$\partial_{\mathbf{j}}[\hat{\mathbf{x}}](f) \;\equiv\; 0 \quad \text{for all } f \text{ whenever } \mathbf{j} \not\geq 0 \tag{6}$$

throughout this paper. A linear combination $c = c_{\mathbf{j}_1}\partial_{\mathbf{j}_1}[\hat{\mathbf{x}}] + \cdots + c_{\mathbf{j}_k}\partial_{\mathbf{j}_k}[\hat{\mathbf{x}}]$ is called a *differential functional*, which will produce a set of numbers $c(F) = \{c(f_1), \ldots, c(f_t)\}$ when applied to the system $F = \{f_1, \ldots, f_t\}$. For differential functionals, the linear anti-differentiation transformation ϕ_i is defined by $\phi_i\big(\sum_{\mathbf{j}} c_{\mathbf{j}}\partial_{\mathbf{j}}[\hat{\mathbf{x}}]\big) = \sum_{\mathbf{j}} c_{\mathbf{j}}\phi_i\big(\partial_{\mathbf{j}}[\hat{\mathbf{x}}]\big)$ with

$$\phi_i\big(\partial_{j_1\ldots j_s}[\hat{\mathbf{x}}]\big) \;=\; \partial_{j'_1\ldots j'_s}[\hat{\mathbf{x}}] \qquad \text{where} \qquad j'_\sigma \;=\; \begin{cases} j_i & \text{if } \sigma \neq i \\ j_i - 1 & \text{if } \sigma = i \end{cases} \tag{7}$$

for $i = 1, \ldots, s$. From (6), we have $\phi_i(\partial_{\mathbf{j}}[\hat{\mathbf{x}}]) = 0$ if $j_i = 0$. With these differential functionals and the linear transformations, we now formulate the multiplicity at a zero $\hat{\mathbf{x}}$ of the nonlinear system (1) as follows.

Definition 1 *Let $F = \{f_1, \ldots, f_t\}$ be a system of functions having derivatives of order $\gamma \geq 1$ at a zero $\hat{\mathbf{x}} \in \mathbb{C}^s$. Let $\mathcal{D}^0_{\hat{\mathbf{x}}}(F) = span\{\partial_{0\ldots0}\}$ and*

$$\mathcal{D}^\alpha_{\hat{\mathbf{x}}}(F) \;=\; \Big\{ c = \sum_{\mathbf{j}\in\mathbb{N}^s,\, c_{\mathbf{j}}\in\mathbb{C},\, |\mathbf{j}|\leq\alpha} c_{\mathbf{j}}\partial_{\mathbf{j}}[\hat{\mathbf{x}}] \;\Big|\; c(F) = \{0\},\ \phi_i(c) \in \mathcal{D}^{\alpha\text{-}1}_{\hat{\mathbf{x}}}(F),\quad \forall\ i = 1, \ldots, s \Big\} \tag{8}$$

for $\alpha = 1, 2, \ldots, \gamma$. We shall call such sets dual subspaces. If $\mathcal{D}^\gamma_{\hat{\mathbf{x}}}(F) = \mathcal{D}^{\gamma\text{-}1}_{\hat{\mathbf{x}}}(F)$, then the vector space

$$\mathcal{D}_{\hat{\mathbf{x}}}(F) \;=\; \bigcup_{\alpha=0}^{\gamma\text{-}1} \mathcal{D}^\alpha_{\hat{\mathbf{x}}}(F) \;\equiv\; \mathcal{D}^\gamma_{\hat{\mathbf{x}}}(F) \tag{9}$$

is called the **dual space** *of the system F at $\hat{\mathbf{x}}$. The dimension $dim\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big)$ is called the* **multiplicity** *of F at $\hat{\mathbf{x}}$.*

Notice that those dual subspaces $\mathcal{D}^\alpha_{\hat{\mathbf{x}}}(F)$'s strictly enlarge as the differential order α increases before reaching certain $\alpha = \mu$. Namely

$$\mathcal{D}^0_{\hat{\mathbf{x}}}(F) \;\subsetneq\; \mathcal{D}^1_{\hat{\mathbf{x}}}(F) \;\subsetneq\; \cdots \;\subsetneq \mathcal{D}^\delta_{\hat{\mathbf{x}}}(F) \;=\; \mathcal{D}^{\delta+1}_{\hat{\mathbf{x}}}(F) \;=\; \cdots \;=\; \mathcal{D}^\gamma_{\hat{\mathbf{x}}}(F) \;=\; \mathcal{D}_{\hat{\mathbf{x}}}(F).$$

The integer δ, called the *depth* which will be defined later, is the highest order of differential functionals in the dual space.

We may also denote the dual space as $\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})$ when the nonlinear system is represented as a mapping $\mathbf{f} = [f_1, \ldots, f_t]^\top$. It is important to note that vanishing at the system $c(F) = \{0\}$ is insufficient for the functional c to be in the dual space $\mathcal{D}_{\hat{\mathbf{x}}}(F)$. This becomes more transparent in single equation $f(x) = 0$ where the multiplicity is *not* the number of vanishing derivatives $f^{(k)}(x) = 0$ at a zero x_*. For instance, infinite number of functionals $\partial_0[0]$, $\partial_2[0]$, $\partial_4[0]$, $\ldots$ vanish at the 1×1-system $\{\sin x\}$, since derivatives $\sin^{(2k)} 0 = 0$ for all integers $k \geq 0$. Among these functionals, however, only $\partial_0[0] \in \mathcal{D}_0(\{\sin x\})$ since

$$\phi_1(\partial_{2k}[0])(\sin x) \;=\; \partial_{2k-1}[0](\sin x) \;=\; \tfrac{(-1)^{k-1}}{(2k-1)!}\cos 0 \;\neq\; 0,$$

namely $\partial_{2k}[0] \notin \mathcal{D}_0(\{\sin x\})$ for all $k \geq 1$, and therefore the multiplicity of $\sin x$ is one at $x = 0$. The crucial *closedness condition*

$$\phi_i(c) \;\in\; \mathcal{D}_{\hat{\mathbf{x}}}(F) \quad \text{for all} \quad c \in \mathcal{D}_{\hat{\mathbf{x}}}(F) \;\text{ and }\; i = 1, \ldots, s \tag{10}$$

in Definition 1 requires the dual space $\mathcal{D}_{\hat{\mathbf{x}}}(F)$ to be invariant under the anti-differentiation transformation ϕ_i's. The following lemma is a direct consequence of the closedness condition.

Lemma 1 *A differential functional* c *is in the dual space* $\mathcal{D}_{\hat{\mathbf{x}}}(F)$ *of the nonlinear system* $F = \{f_1, \ldots, f_t\}$ *at the zero* $\hat{\mathbf{x}}$ *if and only if*

$$c\big((\mathbf{x}-\hat{\mathbf{x}})^{\mathbf{j}} f_i(\mathbf{x})\big) \;=\; 0 \quad \textit{for any} \quad i \in \{1,\ldots,t\} \quad \textit{and} \quad \mathbf{j} \in \mathbb{N}^s \quad \textit{with} \quad \mathbf{j} \geq 0. \tag{11}$$

Proof. For any $\mathbf{j} = (j_1, \ldots, j_s)$, $\mathbf{k} = (k_1, \ldots, k_s)$, and function f, the Leibniz rule of derivatives yields

$$\partial_{\mathbf{j}}[\hat{\mathbf{x}}]\big((\mathbf{x}-\hat{\mathbf{x}})^{\mathbf{k}} f(\mathbf{x})\big) \;=\; \partial_{\mathbf{j}\text{-}\mathbf{k}}[\hat{\mathbf{x}}](f) \;\equiv\; \big(\phi_1^{k_1} \circ \phi_2^{k_2} \circ \cdots \circ \phi_s^{k_s}\big)(\partial_{\mathbf{j}}[\hat{\mathbf{x}}])(f). \tag{12}$$

The equation (11) holds because of the closedness condition (10) and the linearity of c. □

The dual space $\mathcal{D}_{\hat{\mathbf{x}}}(F)$ itself actually contains more structural invariants of the multiple zero beyond the multiplicity for the system F. Via dual subspaces $\mathcal{D}^{\alpha}_{\hat{\mathbf{x}}}(F)$, a *Hilbert function* $\mathrm{h} : \mathbb{N} \to \mathbb{N}$ can be defined as follows:

$$\begin{cases} \mathrm{h}(0) \;=\; \dim\big(\mathcal{D}^0_{\hat{\mathbf{x}}}(F)\big) \equiv 1 \\ \mathrm{h}(\alpha) \;=\; \dim\big(\mathcal{D}^{\alpha}_{\hat{\mathbf{x}}}(F)\big) - \dim\big(\mathcal{D}^{\alpha-1}_{\hat{\mathbf{x}}}(F)\big) \quad \text{for } \alpha \in \{1,2,\ldots\}. \end{cases} \tag{13}$$

The algebraic meaning of this Hilbert function will be elaborated in §4. This Hilbert function is often expressed as a infinite sequence $\{h(0), h(1), \ldots\}$, with which we introduce the *breadth* and the *depth* of dual space $\mathcal{D}_{\hat{\mathbf{x}}}(F)$, denoted by $\beta_{\hat{\mathbf{x}}}(F)$ and $\delta_{\hat{\mathbf{x}}}(F)$ respectively, as follows:

$$\beta_{\hat{\mathbf{x}}}(F) \;=\; \mathrm{h}(1) \quad \text{and} \quad \delta_{\hat{\mathbf{x}}}(F) \;=\; \max\{\,\alpha \mid \mathrm{h}(\alpha) > 0\,\}.$$

In other words, the breadth is the nullity of the Jacobian at $\hat{\mathbf{x}}$ for the system (1) and the depth is the highest differential order of functionals in $\mathcal{D}_{\hat{\mathbf{x}}}(F)$. They are important components of the multiplicity structure that dictate the deflation process for accurate computation of the multiple zero (c.f. §3)

In contrast to the system (3), the system $\{x_1^2 \sin x_1,\ x_2^2 - x_2^2 \cos x_2\}$ also has a zero $(0,0)$ of multiplicity 12 but a different Hilbert function $\{1,2,3,3,2,1,0,\cdots\}$ and a dual space spanned by

$$\overbrace{\partial_{00}}^{1},\ \overbrace{\partial_{10},\ \partial_{01}}^{2},\ \overbrace{\partial_{20},\ \partial_{11},\ \partial_{02}}^{3},\ \overbrace{\partial_{21},\ \partial_{12},\ \partial_{03}}^{3},\ \overbrace{\partial_{13},\ \partial_{22}}^{2},\ \overbrace{\partial_{23}}^{1} \tag{14}$$

The polynomial system $\{x_2^3,\ x_2 - x_3^2,\ x_3 - x_1^2\}$ at origin is again 12-fold with Hilbert function $\{1,\cdots,1,0,\cdots\}$ and a dual space basis

$$\begin{aligned} &\overbrace{\partial_{000}}^{1},\ \overbrace{\partial_{100}}^{1},\ \overbrace{\partial_{200}+\partial_{001}}^{1},\ \cdots,\ \overbrace{\partial_{400}+\partial_{201}+\partial_{002}+\partial_{010}}^{1}, \\ &\cdots,\ \overbrace{\partial_{800}+\partial_{601}+\partial_{402}+\partial_{203}+\partial_{410}+\partial_{004}+\partial_{211}+\partial_{012}+\partial_{020}}^{1} \\ &\cdots,\ \overbrace{\partial_{11,00}+\partial_{901}+\partial_{702}+\partial_{710}+\partial_{503}+\partial_{511}+\partial_{304}+\partial_{312}+\partial_{105}+\partial_{320}+\partial_{113}+\partial_{121}}^{1}. \end{aligned} \tag{15}$$

The last example is of special interest because, as a breadth-one case, its dual space can be computed via a simple recursive algorithm (c.f. §2.3). The dual bases in (14) and (15) are calculated by applying the algorithm NonlinearSystemMultiplicity provided in §2.3 and implemented in ApaTools [28].

We now provide justifications for our multiplicity formulation in Definition 1 by its basic properties. First of all, the multiplicity is a direct generalization of the multiplicity (2) of univariate functions, where the dual space is

$$\mathcal{D}_{x_*}(f) \quad = \quad span\{\partial_0[x_*],\ \partial_1[x_*],\ \ \ldots,\ \partial_{m\text{-}1}[x_*]\}$$

with Hilbert function $\{1, 1, \ldots, 1, 0, \ldots\}$ as well as breadth one and depth $m-1$. Secondly, the multiplicity is well defined for analytic systems as a finite positive integer at any isolated zero $\hat{\mathbf{x}}$, as asserted by the Local Finiteness Theorem below. Namely, the process of calculating the multiplicity will always terminate at certain γ when $\mathcal{D}^{\gamma}_{\hat{\mathbf{x}}}(F) = \mathcal{D}^{\gamma\text{-}1}_{\hat{\mathbf{x}}}(F)$. The dual subspace dimensions $dim\big(\mathcal{D}^0_{\hat{\mathbf{x}}}(F)\big) \le dim\big(\mathcal{D}^1_{\hat{\mathbf{x}}}(F)\big) \le dim\big(\mathcal{D}^2_{\hat{\mathbf{x}}}(F)\big) \le \cdots$ can be unbounded if the zero is in a higher dimensional set of zeros. For example, the dual subspaces $\mathcal{D}^{\alpha}_{(0,0)}(\{\sin(x^2),\ x\cos(y)\})$ never stop expanding since infinitely many linearly independent functionals $\partial_y[(0,0)]$, $\partial_{y^2}[(0,0)]$, $\partial_{y^3}[(0,0)]$, $\ldots$ satisfy the closedness condition and vanish at the zero $(0,0)$. The reason is that $(0,0)$ is not an isolated zero of the system $\{\sin(x^2),\ x\cos y\}$ for which the zero set $\{(0,y)\}$ is the entire y-axis.

Theorem 1 (Local Finiteness Theorem) *Let $F = \{f_1, \ldots, f_t\}$ be a system of analytic functions of s variables $\mathbf{x} = (x_1, \ldots, x_s)$ in an open set $\mathcal{U} \subset \mathbb{C}^s$ and $\hat{\mathbf{x}} \in \mathcal{U}$ is a zero of F. Then $\hat{\mathbf{x}}$ is an isolated zero of F if and only if*

$$\sup_{\alpha \ge 0} dim\big(\mathcal{D}^{\alpha}_{\hat{\mathbf{x}}}(F)\big) \quad < \quad \infty$$

so that the multiplicity $dim\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big)$ is well defined at an isolated zero.

This theorem ensures the multiplicity computation at an isolated zero will terminate in finitely many steps. It also provides a mechanism for identifying nonisolated zeros [2].

When the nonlinear system P consists of polynomials $p_1, \ldots, p_t$ in the variables $x_1, \ldots, x_s$, the multiplicity theory, i.e. the *intersection multiplicity* at a zero of such a special system, has been well studied in algebraic geometry. The following theorem asserts that the multiplicity $dim\big(\mathcal{D}_{\hat{\mathbf{x}}}(P)\big)$ formulated in Definition 1 in this special case is identical to the intersection multiplicity of polynomial systems in algebraic geometry.

Theorem 2 (Multiplicity Consistency Theorem) *Let $P = \{p_1, \ldots, p_t\}$ be a system of polynomials in the variables $x_1, \ldots, x_s$ with complex coefficients. Then the multiplicity $dim\big(\mathcal{D}_{\hat{\mathbf{x}}}(P)\big)$ defined in Definition* 1 *is identical to the intersection multiplicity of the system P at the isolated zero $\hat{\mathbf{x}}$.*

The following Perturbation Invariance Theorem asserts that the multiplicity as defined equals to the number of zeros "multiplied" from a multiple zero when the system is perturbed. As a result, Definition 1 is intuitively justified.

Theorem 3 (Perturbation Invariance Theorem) *Let $F = \{f_1, \ldots, f_s\}$ be a system of functions that are analytic in a neighborhood Ω of an isolated zero $\hat{\mathbf{x}} \in \mathbb{C}^s$ with multiplicity $m = dim\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big)$. Assume $\hat{\mathbf{x}}$ is the unique zero of F in Ω. Then, for any functions $g_1, \ldots, g_s$ that are analytic in Ω, there exists a $\theta > 0$ for which the perturbed system*

$F_\varepsilon = \{f_1 + \varepsilon g_1, \ldots, f_s + \varepsilon g_s\}$ *has exactly* m *zeros in* Ω *counting multiplicities for all* $0 < \varepsilon < \theta$. *More precisely, assuming* $\{\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_k\} = F_\varepsilon^{-1}(0) \cap \Omega$ *are the zeros of* F_ε *in* Ω, *we have*

$$m \;=\; \mathit{dim}\big(\mathcal{D}_{\tilde{\mathbf{x}}}(F)\big) \;=\; \mathit{dim}\big(\mathcal{D}_{\tilde{\mathbf{x}}_1}(F_\varepsilon)\big) + \cdots + \mathit{dim}\big(\mathcal{D}_{\tilde{\mathbf{x}}_k}(F_\varepsilon)\big).$$

We may illustrate this theorem by a computing experiment on the following example.

Example 1 Consider the system $F = \{\sin x \cos y - x,\ \sin y \sin^2 x - y^2\}$ having multiplicity 6 at the zero $(0,0)$. In a small neighborhood of $(0,0)$, we compute the zeros of the perturbed system

$$F_\epsilon \;=\; \{\sin x \cos y - x - \epsilon,\ \sin y \sin^2 x - y^2 + \epsilon\} \tag{16}$$

for small values of ϵ. A cluster of exactly 6 zeros of F_ϵ near $(0,0)$ are found by Newton's iteration using zeros of the truncated Taylor series of F_ϵ as the initial iterates, matching the multiplicity of the system F at $(0,0)$. Table 1 shows the zeros of F_ϵ for $\epsilon = 10^{-8}$ and 10^{-12}. The cluster as shown shrinks to $(0,0)$ when the perturbation decreases in magnitude. □

$\epsilon \;=\; 10^{-8}$	
$\mathbf{x}_1,\ \mathbf{x}_2$	$(-0.0039173928 \mp 0.0000003908\,i,\ -0.0000076728 \pm 0.0000997037\,i)$
$\mathbf{x}_3,\ \mathbf{x}_4$	$(\ 0.0019584003 \pm 0.0033883580\,i,\ \ 0.0000035695 \pm 0.0000935115\,i)$
$\mathbf{x}_5,\ \mathbf{x}_6$	$(\ 0.0019590795 \mp 0.0033879671\,i,\ \ 0.0000040733 \pm 0.0001067848\,i)$
$\epsilon \;=\; 10^{-12}$	
$\mathbf{x}_1,\ \mathbf{x}_2$	$(-0.000181717560 \mp 0.000000000182\,i,\ -0.000000016511 \pm 0.000000999864\,i)$
$\mathbf{x}_3,\ \mathbf{x}_4$	$(\ 0.000090858627 \pm 0.000157362584\,i,\ \ 0.000000008136 \pm 0.000000985770\,i)$
$\mathbf{x}_5,\ \mathbf{x}_6$	$(\ 0.000090858942 \mp 0.000157362403\,i,\ \ 0.000000008372 \pm 0.000001014366\,i)$

Table 1: Zeros of the perturbed system F_ϵ in (16) near $(0,0)$ for $\epsilon = 10^{-8}$ and 10^{-12}.

The proofs of the above three fundamental theorems on multiplicities will be deferred to §4, in which the algebraic foundation of the multiplicity will be established.

Remark on the history of multiplicity: A discussion on the history of the multiplicity formulations for a polynomial system at a zero is given in [6, p.127] from an algebraic geometric point of view. As Fulton points out there have been a great many differing concepts about multiplicity. Mathematicians who have worked on this include Newton, Leibniz, Euler, Cayley, Schubert, Salmon, Kronecker and Hilbert. The dual space approach was first formulated by Macaulay [14] in 1916 for polynomial ideals. Samuel developed this viewpoint with his *Characteristic functions and polynomials* now called *Hilbert functions and polynomials*. More than the multiplicity at a zero of a polynomial system he defines the multiplicity of an arbitrary local ring [26, Ch. VIII §10], which, in the case of a 0-dimensional local ring, is just the sum of the Hilbert function values as in Corollary 1. As we show in §4, this multiplicity is also the $\mathbb{C}$-dimension of the local ring which is now generally accepted as the standard definition of multiplicity in commutative algebra for isolated zeros of systems of equations, see Chapter 4 of [4] for a discussion similar to that of this paper. Symbolic computation of *Gröbner duality* on polynomial ideals was initiated by Marinari, Mora and Möller [15], as well as Mourrain [17]. Stetter and Thallinger introduced numerical computation of the dual basis for an polynomial ideal in [21, 24] and in Stetter's book [22]. Other computational algorithms on the multiplicity problem have recently been proposed in [1], [9], [12], [25], and [29], etc. □

2.2 The Macaulay matrices

Based on the multiplicity formulation, computing the multiplicity structure can be converted to the problem of identifying matrix ranks and kernels. Consider the dual subspace $\mathcal{D}^{\alpha}_{\hat{\mathbf{x}}}(F)$ as defined in (8) for the nonlinear system $F = \{f_1, \ldots, f_t\}$ in $s \leq t$ variables $\mathbf{x} = (x_1, \ldots, x_s)$. By Lemma 1, a functional $c = \sum_{|\mathbf{j}|\leq\alpha} c_{\mathbf{j}}\, \partial_{\mathbf{j}}[\hat{\mathbf{x}}]$ is in the dual subspace $\mathcal{D}^{\alpha}_{\hat{\mathbf{x}}}(F)$ if and only if

$$c\big((\mathbf{x}-\hat{\mathbf{x}})^{\mathbf{k}} f_i(\mathbf{x})\big) \;\equiv\; \sum_{|\mathbf{j}|\leq\alpha} c_{\mathbf{j}} \cdot \partial_{\mathbf{j}}[\hat{\mathbf{x}}]\big((\mathbf{x}-\hat{\mathbf{x}})^{\mathbf{k}} f_i(\mathbf{x})\big) \;=\; 0 \tag{17}$$

for all $|\mathbf{k}| \leq \alpha - 1$ and $i \in \{1, \ldots, s\}$. By a proper ordering of indices $\mathbf{j}$ and $(\mathbf{k}, i)$, equation (17) can be written in matrix form

$$S_\alpha\, \mathbf{c} \;=\; \mathbf{0} \tag{18}$$

where $\mathbf{c}$ is the vector formed by ordering $c_{\mathbf{j}}$ in (17) for $\mathbf{j} \in \mathbb{N}^s$, $\mathbf{j} \geq 0$ and $|\mathbf{j}| \leq \alpha$. The equation (18) determines the dual subspace $\mathcal{D}^{\alpha}_{\hat{\mathbf{x}}}(F)$ that is naturally isomorphic to the kernel $\mathcal{K}(S_\alpha)$ of the matrix S_α, which we call the α-th order Macaulay matrix.

To construct the Macaulay matrices, we choose the *negative degree lexicographical ordering* [8], denoted by $\prec$, on the index set $\mathbb{I}_\alpha \equiv \big\{\, \mathbf{j} \in \mathbb{N}^s \;\big|\; \mathbf{j} \geq 0,\ |\mathbf{j}| \leq \alpha \,\big\}$:

$$\begin{aligned} \mathbf{i} \;\prec\, \mathbf{j} \quad & \text{if} \quad |\mathbf{i}| < |\mathbf{j}|, \quad \text{or} \\ & (|\mathbf{i}| = |\mathbf{j}| \quad \text{and} \quad \exists\ 1 \leq \sigma \leq s: \quad i_1 = j_1, \ldots,\ i_{\sigma\text{-}1} = j_{\sigma\text{-}1},\ i_\sigma < j_\sigma). \end{aligned}$$

The Macaulay matrix S_α is of size $m_\alpha \times n_\alpha$ where

$$m_\alpha \;=\; \begin{pmatrix} \alpha - 1 + s \\ \alpha - 1 \end{pmatrix} \quad \text{and} \quad n_\alpha = \begin{pmatrix} \alpha + s \\ \alpha \end{pmatrix}.$$

We view the rows to be indexed by $(\mathbf{x}-\hat{\mathbf{x}})^{\mathbf{k}} f_i$ for $(\mathbf{k}, i) \in \mathbb{I}_{\alpha-1} \times \{1, \cdots, t\}$ with ordering $(\mathbf{k}, i) \prec (\mathbf{k}', i')$ if $\mathbf{k} \prec \mathbf{k}'$ in $\mathbb{I}_{\alpha-1}$ or $\mathbf{k} = \mathbf{k}'$ but $i < i'$, and the columns are indexed by the differential functionals $\partial_{\mathbf{j}}$ for $\mathbf{j} \in \mathbb{I}_\alpha$. The entry of S_α, at the intersection of the row and column indexed by $(\mathbf{x}-\hat{\mathbf{x}})^{\mathbf{k}} f_i$ and $\partial_{\mathbf{j}}$ respectively, is the value of $\partial_{\mathbf{j}}[\hat{\mathbf{x}}]\big((\mathbf{x}-\hat{\mathbf{x}})^{\mathbf{k}} f_i\big)$. With this arrangement, S_α is the upper-left $m_\alpha \times n_\alpha$ submatrix of subsequent Macaulay matrices S_σ, for $\sigma \geq \alpha$, as illustrated in Example 2. The following corollary is thus straightforward.

Corollary 1 *Let $F = \{f_1, \ldots, f_t\}$ be a nonlinear system of functions in variables $\mathbf{x} = (x_1, \ldots, x_s)$ with a zero $\hat{\mathbf{x}}$. Then for each $\alpha > 0$, the dual subspace $\mathcal{D}^{\alpha}_{\hat{\mathbf{x}}}(\mathcal{I})$ is isomorphic to the kernel $\mathcal{K}(S_\alpha)$ of the Macaulay matrix S_α. In particular, with $S_0 \equiv [f_1(\hat{\mathbf{x}}), \ldots, f_t(\hat{\mathbf{x}})]^\top = \mathbf{0}$, the Hilbert function*

$$\mathrm{h}(\alpha) \;=\; \textit{nullity}\,(\,S_\alpha\,) - \textit{nullity}\,(\,S_{\alpha\text{-}1}\,) \tag{19}$$

for $\alpha = 1, 2, \cdots$.

Notice that for an obvious ordering $\prec$ of $\mathbb{I}_1$, we can arrange

$$S_1 \;=\; \left[\begin{array}{c|c} f_1(\hat{\mathbf{x}}) & \\ \vdots & J(\hat{\mathbf{x}}) \\ f_t(\hat{\mathbf{x}}) & \end{array}\right] \;\equiv\; \left[\begin{array}{c|c} 0 & \\ \vdots & J(\hat{\mathbf{x}}) \\ 0 & \end{array}\right] \tag{20}$$

where $J(\hat{\mathbf{x}})$ is the Jacobian of the system $\{f_1, \ldots, f_t\}$ at $\hat{\mathbf{x}}$.

Example 2 Consider the system $F = \{x_1 - x_2 + x_1^2,\ x_1 - x_2 + x_2^2\}$ at $\hat{\mathbf{x}} = (0,0)$. Figure 2 shows the expansion of the Macaulay matrices from S_1 to S_2, then S_3, with rows and columns labeled by $\mathbf{x}^{\mathbf{k}} f_i$ and $\partial_{\mathbf{j}}$ respectively. The table beneath the Macaulay matrices in Figure 2 shows the bases for the kernels as row vectors using the same column indices. It is instructive to compare this pair of arrays to those in [14, § 65] or the reconstruction of Macaulay's arrays in [16, Example 30.4.1].

The kernels can be converted to bases of dual subspaces using the indices in the table:

$$\begin{aligned}
\mathcal{D}^0_{(0,0)}(F) &= \mathit{span}\{\partial_{00}\} \\
\mathcal{D}^1_{(0,0)}(F) &= \mathit{span}\{\partial_{00},\ \ \partial_{10} + \partial_{01}\} \\
\mathcal{D}^2_{(0,0)}(F) &= \mathit{span}\{\partial_{00},\ \ \partial_{10} + \partial_{01},\ \ -\partial_{10} + \partial_{20} + \partial_{11} + \partial_{02}\}.
\end{aligned}$$

It is easy to verify that $\mathit{nullity}(S_3) = \mathit{nullity}(S_2) = 3$. Therefore, the Hilbert function $\mathrm{h}(\mathbb{N}) = \{1,1,1,0,\cdots\}$. The multiplicity equals 3. The dual space $\mathcal{D}_{(0,0)}(F) = \mathcal{D}^2_{(0,0)}(F)$ with breadth $\beta_{(0,0)}(F) = \mathrm{h}(1) = 1$ and depth $\delta_{(0,0)}(F) = \max\{\alpha \,|\, \mathrm{h}(\alpha) > 0\} = 2$. The complete multiplicity structure is in order. □

multiplicity		$\lvert\mathbf{j}\rvert=0$	$\lvert\mathbf{j}\rvert=1$		$\lvert\mathbf{j}\rvert=2$			$\lvert\mathbf{j}\rvert=3$			
matrices ↘		∂_{00}	∂_{10}	∂_{01}	∂_{20}	∂_{11}	∂_{02}	∂_{30}	∂_{21}	∂_{12}	∂_{03}
S_1	$\lvert\mathbf{k}\rvert=0$ f_1	0	1	−1	1	0	0	0	0	0	0
	f_2	0	1	−1	0	0	1	0	0	0	0
S_2	$\lvert\mathbf{k}\rvert=1$ x_1f_1	0	0	0	1	−1	0	1	0	0	0
	x_1f_2	0	0	0	1	−1	0	0	0	1	0
	x_2f_1	0	0	0	0	1	−1	0	1	0	0
	x_2f_2	0	0	0	0	1	−1	0	0	0	1
S_3	$\lvert\mathbf{k}\rvert=2$ $x_1^2f_1$	0	0	0	0	0	0	1	−1	0	0
	$x_1^2f_2$	0	0	0	0	0	0	1	−1	0	0
	$x_1x_2f_1$	0	0	0	0	0	0	0	1	−1	0
	$x_1x_2f_2$	0	0	0	0	0	0	0	1	−1	0
	$x_2^2f_1$	0	0	0	0	0	0	0	0	1	−1
	$x_2^2f_2$	0	0	0	0	0	0	0	0	1	−1

bases for kernels (transposed as row vectors)

$\mathcal{K}(S_0)$	1	0	0	0	0	0	0	0	0	0
$\mathcal{K}(S_1)$	0	1	1	0	0	0	0	0	0	0
$\mathcal{K}(S_2)$	0	−1	0	1	1	1	0	0	0	0
$\mathcal{K}(S_3)$										

Figure 2: Expansion of the Macaulay matrices for the polynomial system in Example 2

By identifying the multiplicity structure of a nonlinear system with the kernels and nullities of Macaulay matrices, the multiplicity computation can be reliably carried out by matrix rank-revealing, as we shall elaborate in §2.3.

2.3 Computing the multiplicity structure

The multiplicity as well as the multiplicity structure can be computed using symbolic, symbolic-numeric or floating point computation based on Corollary 1. The main algorithm can be outlined in the following pseudo-code.

Algorithm: NonlinearSystemMultiplicity

Input: system $F = \{f_1, \cdots, f_t\}$ and zero $\hat{\mathbf{x}} \in \mathbb{C}^s$

- initialize $S_0 = O_{t\times 1}$, $\mathcal{K}(S_0) = \mathit{span}\{[1]\}$, $\mathrm{h}(0) = 1$
- for $\alpha = 1, 2, \cdots$ do
  - expand $S_{\alpha\text{-}1}$ to S_α, and embed $\mathcal{K}(S_{\alpha\text{-}1})$ into $\mathcal{K}(S_\alpha)$
  - find $\mathcal{K}(S_\alpha)$ by expanding $\mathcal{K}(S_{\alpha\text{-}1})$
  - if $\mathit{nullity}(S_\alpha) = \mathit{nullity}(S_{\alpha\text{-}1})$ then
    $\delta = \alpha - 1$, $\mathrm{h}(\alpha) = 0$, break the loop
    otherwise, get $\mathrm{h}(\alpha)$ by (19)
  - end if
- end do
- convert $\mathcal{K}(S_\delta)$ to $\mathcal{D}_{\hat{\mathbf{x}}}(F)$

Output: multiplicity $m = \sum_\alpha \mathrm{h}(\alpha)$, the Hilbert function h, $\mathcal{D}_{\hat{\mathbf{x}}}(F)$ basis, depth $\delta_{\hat{\mathbf{x}}}(F)$, and breadth $\beta_{\hat{\mathbf{x}}}(F) = \mathrm{h}(1)$

This algorithm turns out to be essentially equivalent to Macaulay's procedure of 1916 for finding *inverse arrays* of *dialytic arrays* [14, 16], except that Macaulay's algorithm requires construction of dialytic arrays with full row rank. This requirement is difficult and costly to implement with approximate systems or the approximate zeros.

Implementation of the algorithm NonlinearSystemMultiplicity is straightforward for symbolic computation when the system and zero are exact and properly represented. Applying this multiplicity-finding procedure on approximate zeros and/or inexact systems requires the notions and algorithms of numerical rank-revealing at the step "find $\mathcal{K}(S_\alpha)$" in Algorithm NonlinearSystemMultiplicity.

The *numerical rank* of a matrix A is defined as the minimum rank of matrices within a threshold θ [7, §2.5.5]:

$$\mathit{rank}_\theta(A) = \min_{\|A-B\|_2 \le \theta} \mathit{rank}(B).$$

The *numerical kernel* $\mathcal{K}_\theta(A)$ of A is the (exact) kernel $\mathcal{K}(B)$ of B that is nearest to A with $\mathit{rank}(B) = \mathit{rank}_\theta(A)$. With this reformulation, numerical rank/kernel computation becomes well-posed. We refer to [13] for details.

Numerical rank-revealing applies the iteration [13]

$$\begin{cases} \mathbf{u}_{k+1} = \mathbf{u}_k - \begin{bmatrix} 2\|A\|_\infty \mathbf{u}_k \\ A \end{bmatrix}^\dagger \begin{bmatrix} \|A\|_\infty(\mathbf{u}_k^{\mathsf{H}}\mathbf{u}_k - 1) \\ A\mathbf{u}_k \end{bmatrix} \\ \varsigma_{k+1} = \dfrac{\|A\mathbf{u}_{k+1}\|_2}{\|\mathbf{u}_{k+1}\|_2}, \quad k = 0, 1, \cdots \end{cases} \tag{21}$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose inverse. From a randomly chosen $\mathbf{u}_0$, this iteration virtually guarantees convergence to a numerical null vector $\mathbf{u}$, and $\{\varsigma_k\}$ will converge to the distance ς between A and the nearest rank-deficient matrix.

With a numerical null vector $\mathbf{u}$ available, applying (21) on $\hat{A} = \begin{bmatrix} \|A\|_\infty \mathbf{u}^{\mathsf{H}} \\ A \end{bmatrix}$ yields another sequence $\{\hat{\mathbf{u}}_k\}$ that converges to a numerical null vector $\mathbf{v}$ of A orthogonal to $\mathbf{u}$, and the scalar sequence

$\{\hat{\varsigma}_k\}$ converges to the distance between A and the nearest matrix with nullity 2. This process can be recursively continued by stacking $\|A\|_\infty \mathbf{v}^{\mathsf{H}}$ on top of $\hat{A}$ and applying (21) on the new stacked matrix.

We now describe the numerical procedure that carries out the step of computing $\mathcal{K}(S_\alpha)$ in Algorithm NonlinearSystemMultiplicity.

The kernel $\mathcal{K}_\theta(S_0) = span\{[1]\}$. Assume an orthonormal basis $Y = \left[\mathbf{y}_1, \cdots, \mathbf{y}_\mu\right]$ for $\mathcal{K}_\theta(S_{\alpha\text{-}1})$ and the QR decomposition $\begin{bmatrix} TY^{\mathsf{H}} \\ S_{\alpha\text{-}1} \end{bmatrix} = Q_{\alpha\text{-}1}\begin{bmatrix} R_{\alpha\text{-}1} \\ O \end{bmatrix}$ are both available, where $Q_{\alpha\text{-}1}$ is unitary, $R_{\alpha\text{-}1}$ is square upper-triangular and T is a diagonal scaling matrix.

Embedding $\mathbf{y}_i$'s into $\mathbb{C}^{n_\alpha}$ by appending zeros at the bottom to form $\mathbf{z}_i$ for $i = 1, \cdots, \mu$, it is clear that the columns of $Z = \left[\mathbf{z}_1, \cdots, \mathbf{z}_\mu\right]$ form a subset of an orthonormal basis for $\mathcal{K}_\theta(S_\alpha)$. Also, we have matrix partitions

$$S_\alpha = \begin{bmatrix} S_{\alpha\text{-}1} & F \\ O & G \end{bmatrix}, \qquad \begin{bmatrix} TZ^{\mathsf{H}} \\ S_\alpha \end{bmatrix} = \begin{bmatrix} TY^{\mathsf{H}} & O \\ S_{\alpha\text{-}1} & F \\ O & G \end{bmatrix} \begin{bmatrix} Q_{\alpha\text{-}1} \begin{bmatrix} R_{\alpha\text{-}1} & F_1 \\ O & F_2 \end{bmatrix} \\ \hline \begin{bmatrix} O & G \end{bmatrix} \end{bmatrix}$$

where $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = Q_{\alpha\text{-}1}^{\mathsf{H}} \begin{bmatrix} O \\ F \end{bmatrix}$. Let $\hat{Q}\begin{bmatrix} \hat{R} \\ O \end{bmatrix} = \begin{bmatrix} F_2 \\ G \end{bmatrix}$ be a QR decomposition. Then

$$\begin{bmatrix} TZ^{\mathsf{H}} \\ S_\alpha \end{bmatrix} = Q_\alpha \begin{bmatrix} R_{\alpha\text{-}1} & F_1 \\ O & \hat{R} \\ O & O \end{bmatrix} = Q_\alpha \begin{bmatrix} R_\alpha \\ O \end{bmatrix} \tag{22}$$

with a proper accumulation of $Q_{\alpha\text{-}1}$ and $\hat{Q}$ into Q_α. This implies

$$\mathcal{K}(R_\alpha) = \mathcal{K}(S_\alpha) \bigcap \mathcal{K}(Z^{\mathsf{H}}) = \mathcal{K}(S_\alpha) \bigcap \mathcal{K}_\theta(S_{\alpha\text{-}1})^\perp .$$

Therefore $\mathcal{K}_\theta(R_\alpha)$ consists of numerical null vectors of S_α that are approximately orthogonal to those of $S_{\alpha\text{-}1}$. The procedure below produces the numerical kernel $\mathcal{K}_\theta(R_\alpha)$.

- `let` $A = R_\alpha$
- `for` $i = 1, 2, \cdots$ `do`
  - `apply iteration (21), stop at` $\mathbf{u}$ `and` ς `with proper criteria`
  - `if` $\varsigma > \theta$, `exit, end if`
  - `get` $\mathbf{z}_{\mu+i} = \mathbf{u}$, `reset` A `with` $\begin{bmatrix} \|A\|_\infty \mathbf{u}^{\mathsf{H}} \\ A \end{bmatrix}$
  - `update the QR decomposition` $A = QR$

  `end for`

Upon exit, vectors $\mathbf{z}_{\mu+1}, \cdots, \mathbf{z}_{\mu+\nu}$ are remaining basis vectors of $\mathcal{K}_\theta(S_\alpha)$ aside from previously obtained $\mathbf{z}_1, \cdots, \mathbf{z}_\mu$. Furthermore, the QR decomposition of $\begin{bmatrix} \hat{T}\hat{Z}^{\mathsf{H}} \\ S_\alpha \end{bmatrix}$ is a by-product from a proper accumulation of orthogonal transformations. Here $\hat{Z} = \left[\mathbf{z}_1, \cdots, \mathbf{z}_{\mu+\nu}\right]$ with a column permutation and $\hat{T}$ is again a scaling matrix.

Algorithm NonlinearSystemMultiplicity is implemented as a function module in the software package ApaTools [28] for Maple. For an isolated zero of a given system along with a rank threshold, the software produces the multiplicity, breadth, depth, Hilbert function, and a basis for the dual space. The software performs symbolic (exact) computation when the rank threshold is set to zero, and carries out numerical computation otherwise.

Example 3 Let F be the system in (29) and the approximate zero $\hat{\mathbf{x}} = (0.9999999999999999, 1.999999999999999, 3.0)$ is accurately calculated using the depth-deflation method (c.f. §3.1). Since the estimated error of the approximate zero is 1.94×10^{-14}, we set the rank threshold to be slightly larger as 10^{-12}. Algorithm NonlinearSystemMultiplicity accurately produces the multiplicity 11, breadth 3, depth 4, Hilbert function $\{1, 3, 3, 3, 1, 0, \ldots, \}$ and the (approximate) dual basis

$$\partial_{000},\ \partial_{100},\ \partial_{010},\ \partial_{001},\ \partial_{200},\ \partial_{020},\ \partial_{002},\ .707106781186544\,\partial_{101} + .707106781186543\,\partial_{030}$$
$$.707106781186544\,\partial_{011} + .707106781186545\,\partial_{300},\ .707106781186545\,\partial_{110} + .707106781186545\,\partial_{003},$$
$$.500000000000008\,\partial_{111} + .500000000000007\,\partial_{400} + .500000000000009\,\partial_{040} + .500000000000008\,\partial_{004}.$$

□

Remarks on computational issues: For an exact system, the accuracy of the zero $\hat{\mathbf{x}}$ can be arbitrarily high using multiprecision or a deflation method described in §3. As a result, numerical rank-revealing with sufficient low threshold will ensure accurate multiplicity identification. For inexact systems, the approximate zeros may carry substantial errors due to the inherent sensitivity. In this case, setting a proper threshold θ for the numerical rank revealing may become difficult. The depth-deflation method given in §3 is effective in calculating the zeros to the highest possible accuracy that may allow accurate identification of the multiplicity. However, there will always be intractable cases. For those systems with obtainable multiplicity structure at an approximate solution, the rank threshold needs to be set by users according to the magnitude of errors on the system and solution. Generally, the threshold should be set higher than the size of error.

The size increase of Macaulay matrices may become a prohibitive obstacle when the number of variables is large, compounding with high depth $\delta_{\hat{\mathbf{x}}}(F)$. Most notably, when the breadth $\beta_{\hat{\mathbf{x}}}(F) = 1$, the depth is maximal $\delta_{\hat{\mathbf{x}}}(F) = m - 1$. In this situation, high order α's and large sizes of Macaulay matrices S_α are inevitable. A special case algorithm BreadthOneMultiplicity in §3.3 is designed to deal with this difficulty. A recently developed closedness subspace strategy [29] improves the efficiency of multiplicity computation substantially by reducing the size of the matrices. □

3 Accurate computation of a multiple zero by deflating its depth

It is well known in textbooks that multiple zeros are highly sensitive to perturbations and are therefore difficult to compute accurately using floating point arithmetic. Even for a single univariate equation $f(x) = 0$, as mentioned before, there is a perceived barrier of "attainable accuracy": The number of attainable digits at a multiple zero is bounded by the hardware precision divided

by the multiplicity. This accuracy barrier is largely erased recently in [27] for univariate polynomial equations. For general nonlinear multivariate systems, we propose a general *depth-deflation* method in this section, as well as its special case variation for breadth one systems for accurate computation of multiple zeros without extending hardware precision even when the given system is perturbed.

3.1 The depth-deflation method

The hypersensitivity in calculating an approximation $\tilde{x}_*$ to an m-fold zero x_* can be illustrated by solving $f(x) = 0$ for $f(x) = x^m$. When the function is perturbed slightly to $f_\varepsilon(x) = x^m - \varepsilon$, the error becomes

$$|\tilde{x}_* - x_*| \quad = \quad |f - f_\varepsilon|^{\frac{1}{m}}.$$

The asymptotic condition number is

$$\sup_{\varepsilon>0} \frac{|\tilde{x}_* - x_*|}{|f - f_\varepsilon|} \quad = \quad \infty$$

when the multiplicity $m > 1$. Consequently, multiple zeros are referred to as "singular" or "infinitely sensitive" to perturbations in the literature. On the other hand, a simple zero is considered "regular" with a finite condition number as stated in the following lemma.

Lemma 2 *Let $\mathbf{f} = [f_1, \ldots, f_t]^{\mathsf{H}}$ be a system of s-variate functions that are twice differentiable in a neighborhood of $\hat{\mathbf{x}} \in \mathbb{C}^s$. If the Jacobian $J(\hat{\mathbf{x}})$ of $\mathbf{f}(\mathbf{x})$ at $\hat{\mathbf{x}}$ is injective, then*

$$\big\|\tilde{\mathbf{x}} - \hat{\mathbf{x}}\big\|_2 \quad \leq \quad \big\|J(\hat{\mathbf{x}})^+\big\|_2 \, \big\|\mathbf{f}(\tilde{\mathbf{x}}) - \mathbf{f}(\hat{\mathbf{x}})\big\|_2 + O\big(\|\mathbf{f}(\tilde{\mathbf{x}}) - \mathbf{f}(\hat{\mathbf{x}})\|_2^2\big) \tag{23}$$

with $\|J(\hat{\mathbf{x}})^+\|_2 < \infty$.

Proof. The injectiveness of $J(\hat{\mathbf{x}})$ implies $t \geq s$ and $\mathit{rank}\big(J(\hat{\mathbf{x}})\big) = s$. Without loss of generality, we assume the submatrix of $J(\hat{\mathbf{x}})$ consists of its first s rows is invertible. By the Inverse Function Theorem, the function $[y_1, \ldots, y_s]^{\mathsf{H}} = [f_1(\mathbf{x}), \ldots, f_s(\mathbf{x})]^{\mathsf{H}}$ has a continuously differentiable inverse $\mathbf{x} = \mathbf{g}(y_1, \ldots, y_s)$ in a neighborhood of $[\hat{y}_1, \ldots, \hat{y}_s]^{\mathsf{H}} = [f_1(\hat{\mathbf{x}}), \ldots, f_s(\hat{\mathbf{x}})]^{\mathsf{H}}$, permitting $\|\mathbf{x} - \hat{\mathbf{x}}\|_2 \leq C\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\hat{\mathbf{x}})\|_2$ for $\mathbf{x}$ in a neighborhood of $\hat{\mathbf{x}}$. Since

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\hat{\mathbf{x}}) \quad = \quad J(\hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{r}(\mathbf{x}) \quad \text{or} \quad \mathbf{x} - \hat{\mathbf{x}} \quad = \quad J(\hat{\mathbf{x}})^+\big[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\hat{\mathbf{x}}) - \mathbf{r}(\mathbf{x})\big]$$

where $\|\mathbf{r}(\mathbf{x})\|_2 = O\big(\|\mathbf{x} - \hat{\mathbf{x}}\|_2^2\big) = O\big(\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\hat{\mathbf{x}})\|_2^2\big)$, we thus have (23). □

In light of Lemma 2, we may define the **condition number**

$$\kappa_{\mathbf{f}}(\hat{\mathbf{x}}) \quad = \quad \begin{cases} \|J(\hat{\mathbf{x}})^+\|_2 & \text{if } J(\hat{\mathbf{x}}) \text{ is injective} \\ \infty & \text{otherwise} \end{cases} \tag{24}$$

of the system $\mathbf{f}$ at a zero $\hat{\mathbf{x}}$ as a sensitivity measurement, with which we can establish an error estimate

$$\|\tilde{\mathbf{x}} - \hat{\mathbf{x}}\|_2 \quad \approx \quad \kappa_{\mathbf{f}}(\tilde{\mathbf{x}}) \cdot \|\mathbf{f}(\tilde{\mathbf{x}})\|_2 \tag{25}$$

of the approximate zero $\tilde{\mathbf{x}}$ using the residual $\|\mathbf{f}(\tilde{\mathbf{x}})\|_2$.

Solving a nonlinear system for a multiple zero is an ill-posed problem. The straightforward Newton's iteration attains only a few correct digits of the zero besides losing its superlinear convergence rate, if it converges at all. Similar to other ill-posed problems, computing a multiple zero for higher accuracy needs a regularization procedure. An effective regularization approach is the *deflation* [11, 18]. For instance, Leykin, Verschelde and Zhao [11] propose a deflation method which successfully restored the quadratic convergence of Newton's iteration. From our perspective, perhaps the most important feature of deflation strategy should reside in transforming an ill-posed zero-finding into a well-posed least squares problem. As a result, the multiple zero can be calculated to high accuracy.

We hereby propose two new versions of the deflation method, one for the general cases and the other for the cases where the breadth of the system is one at the zero, and both versions are called *depth-deflation* methods. We first derive our general depth-deflation method here. The version for breadth-one systems follows in §3.3.

Let $\mathbf{f} : \mathbb{C}^s \longrightarrow \mathbb{C}^t$ represent a nonlinear system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ where $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \cdots, f_t(\mathbf{x})]^\top$, $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{C}^s$ with $t \geq s$, and $\hat{\mathbf{x}}$ be an isolated zero of $\mathbf{f}(\mathbf{x})$. Denote $J(\mathbf{x})$ as the Jacobian of $\mathbf{f}(\mathbf{x})$. If $\hat{\mathbf{x}}$ is a simple zero, then $J(\hat{\mathbf{x}})$ is injective with pseudo-inverse $J(\hat{\mathbf{x}})^+ = [J(\hat{\mathbf{x}})^{\mathsf{H}} J(\hat{\mathbf{x}})]^{-1} J(\hat{\mathbf{x}})^{\mathsf{H}}$, and the Gauss-Newton iteration

$$\mathbf{x}^{(n+1)} \;=\; \mathbf{x}^{(n)} - J(\mathbf{x}^{(n)})^+ \, \mathbf{f}(\mathbf{x}^{(n)}) \quad \text{for} \quad n = 0, 1, \ldots \tag{26}$$

locally converges to $\hat{\mathbf{x}}$ at a quadratic rate. More importantly in this regular case, solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for the solution $\hat{\mathbf{x}}$ is a well-posed problem and the condition number $\|J(\hat{\mathbf{x}})^+\| < \infty$.

When $\hat{\mathbf{x}}$ is a multiple zero of $\mathbf{f}(\hat{\mathbf{x}})$, however, the Jacobian $J(\hat{\mathbf{x}})$ is rank-deficient. In this *singular* case, the zero $\hat{\mathbf{x}}$ is *underdetermined* by the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ because it is also a solution to $J(\mathbf{x})\mathbf{y} = \mathbf{0}$ for some $\mathbf{y} \neq \mathbf{0}$. In order to eliminate the singularity and thus curb the hypersensitivity, perhaps further constraints should be imposed.

Let $n_1 = \textit{nullity}\,(\,J(\hat{\mathbf{x}})\,)$ which is strictly positive at the multiple zero $\hat{\mathbf{x}}$. Denote $\mathbf{x}_1 = \mathbf{x}$ and $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$. Then, for almost all choices of an $n_1 \times s$ random matrix R_1, the matrix $\begin{bmatrix} J(\hat{\mathbf{x}}_1) \\ R_1 \end{bmatrix}$ is of full (column) rank. It is easy to see that the linear system $\begin{bmatrix} J(\hat{\mathbf{x}}_1) \\ R_1 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{bmatrix}$ has a unique solution $\mathbf{x}_2 = \hat{\mathbf{x}}_2 \neq \mathbf{0}$. Here $\mathbf{e}_1$ is the first canonical vector $[1, 0, \ldots, 0]^\top$ of a proper dimension. As a result, a new $(2t + k) \times (2s)$ system

$$\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) \;\equiv\; \begin{bmatrix} \mathbf{f}(\mathbf{x}_1) \\ \begin{bmatrix} J(\mathbf{x}_1) \\ R_1 \end{bmatrix} \mathbf{x}_2 - \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{bmatrix} \end{bmatrix} \tag{27}$$

has an isolated solution $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$.

If $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ is a simple zero of $\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2)$, then the singularity of $\mathbf{f}(x)$ at $\hat{\mathbf{x}}$ is "deflated" by solving $\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) = 0$ for $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ as a well-posed problem using the Gauss-Newton iteration (26) on $\mathbf{f}_1$. However, $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ may still be a multiple zero of $\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2)$ and, in this case, the depth-deflation process must continue. We can simply repeat the depth-deflation method above on $\mathbf{f}_1$. Generally, assume $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_{2^\alpha})$ is still an isolated multiple zero of $\mathbf{f}_\alpha(\mathbf{x}_0, \ldots, \mathbf{x}_{2^\alpha})$ after α steps of depth-deflation with a Jacobian $J_\alpha(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_{2^\alpha})$ of nullity $n_\alpha > 0$. The next depth-deflation

step expands the system to

$$\mathbf{f}_{\alpha+1}(\mathbf{x}_1,\ldots,\mathbf{x}_{2^{\alpha+1}}) \equiv \begin{bmatrix} \mathbf{f}_\alpha(\mathbf{x}_1,\ldots,\mathbf{x}_{2^\alpha}) \\ \begin{bmatrix} J_\alpha(\mathbf{x}_1,\ldots,\mathbf{x}_{2^\alpha}) \\ R_{\alpha+1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{2^\alpha+1} \\ \vdots \\ \mathbf{x}_{2^{\alpha+1}} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{bmatrix} \end{bmatrix} \tag{28}$$

where $R_{\alpha+1}$ is a randomly selected matrix of $n_{\alpha+1}$ rows and the same number of columns as $J_\alpha(\mathbf{x}_1,\ldots,\mathbf{x}_{2^\alpha})$. The depth-deflation process continues recursively by expanding $\mathbf{f}(\mathbf{x}_1)$ to $\mathbf{f}_1(\mathbf{x}_1,\mathbf{x}_2)$, $\mathbf{f}_2(\mathbf{x}_1,\ldots,\mathbf{x}_4)$, ..., until reaching an expanded system $\mathbf{f}_\sigma(\mathbf{x}_1,\ldots,\mathbf{x}_{2^\sigma})$ with an isolated zero $(\hat{\mathbf{x}}_1,\ldots,\hat{\mathbf{x}}_{2^\sigma})$ that is no longer singular. The following Depth Deflation Theorem ensures the depth-deflation process will terminate and the number of deflation steps is bounded by the depth $\delta_{\hat{\mathbf{x}}}(\mathbf{f})$.

Theorem 4 (Depth Deflation Theorem) *Let $\mathbf{f} = [f_1,\cdots,f_t]^\top$ be a system of functions in the variables $\mathbf{x} = (x_1,\cdots,x_s)$ with an isolated multiple zero $\hat{\mathbf{x}}$ of depth $\delta_{\hat{\mathbf{x}}}(\mathbf{f})$. Then there is an integer $\sigma \le \delta_{\hat{\mathbf{x}}}(\mathbf{f})$ such that the depth-deflation process terminates at the expanded system $\mathbf{f}_\sigma(\mathbf{x}_1,\ldots,\mathbf{x}_{2^\sigma})$ with a simple zero $(\hat{\mathbf{x}}_1,\ldots,\hat{\mathbf{x}}_{2^\sigma})$ where $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$. Furthermore, the depth-deflation method generates 2^σ differential functionals in the dual space $\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})$ as by-products.*

We shall prove this Depth Deflation Theorem via multiplicity analysis in §3.2.

For polynomial systems, Leykin, Verschelde and Zhao proved that each deflation step of their method deflates intersection multiplicity by at least one [11, Theorem 3.1]. Theorem 4 improves the deflation bound substantially since the depth is much smaller than the multiplicity when the breath is larger than one. The computing cost increases exponentially as the depth-deflation continues since each depth-deflation step doubles the number of variables. Fortunately, computing experiments suggest that, for a multiple zero of breadth larger than one, very few depth-deflation steps are required. At breadth-one zeros, we shall derive a special case depth-deflation method in §3.3.

The high accuracy achieved by applying the depth-deflation method can be illustrated in the following examples.

Example 4 Consider the system

$$\begin{cases} (x-1)^3 + .416146836547142\,(z-3)\sin y + .909297426825682\,(z-3)\cos y &= 0 \\ (y-2)^3 + .989992496600445\,(x-1)\sin z + .141120008059867\,(x-1)\cos z &= 0 \\ (z-3)^3 - .540302305868140\,(y-2)\sin x + .841470984807897\,(y-2)\cos x &= 0 \end{cases} \tag{29}$$

being a perturbation of magnitude 10^{-15} from an exact system with zero $(1,2,3)$ of multiplicity 11, depth 4 and breadth 3. Using 16-digit arithmetic in Maple to simulate the hardware precision, Newton's iteration without depth-deflation attains only 4 correct digits, while a single depth-deflation step eliminates the singularity and obtains 15 correct digits, as shown in the following table. The error estimates listed in the table are calculated using (25) which provides an adequate accuracy measurement for the computed zeros.

		without deflation	with deflation	exact value
zero	x	1.0003	0.999999999999999	1.0
	y	1.9997	1.999999999999999	2.0
	z	3.0003	3.000000000000000	3.0
error estimate		0.00027	0.000000000000019	

□

Example 5 Consider the system

$$\begin{aligned} e^z - .944956946314738\,\cos y + .327194696796152\,\sin y &= 0 \\ z^2 - y^3 - y^2 - .333333333333333\,y - .0370370370370370 &= 0 \\ y^2 + .666666666666667\,y + .148148148148148 - x^3 + x^2 - .333333333333333\,x &= 0. \end{aligned}$$

This is a perturbation of magnitude 10^{-15} from an exact system with zero $(1/3, -1/3, 0)$ of multiplicity 9, depth 5, breadth 2 and Hilbert function $\{1, 2, 2, 2, 1, 1, 0, \ldots\}$. Again, using 16-digits arithmetic in Maple, Newton's iteration diverges from the initial iterate $(0.31, -0.31, 0.01)$. In contrast, our depth-deflation method takes three deflation steps to eliminate the singularity and obtains 15 correct digits of the multiple zero:

		without deflation	with deflation	exact value
zero	x	diverges	0.3333333333333336	$1/3$
	y	diverges	-0.3333333333333334	$-1/3$
	z	diverges	0.0000000000000002	0
error estimate		—	0.0000000000001950	

□

3.2 Multiplicity analysis of the depth-deflation method

We shall use some additional differential notations and operations. The original variables $\mathbf{x} = [x_1, \cdots, x_s]^\top$ will be denoted by $\mathbf{x}_1$ in accordance with the notation for the auxiliary (vector) variables $\mathbf{x}_2$, $\mathbf{x}_3$ etc. For any fixed or variable vector $\mathbf{y} = [y_1, \cdots, y_s]^\top$, the *directional differentiation operator* along $\mathbf{y}$ is defined as

$$\nabla_{\mathbf{y}} \equiv y_1 \frac{\partial}{\partial x_1} + \cdots + y_s \frac{\partial}{\partial x_s}. \tag{30}$$

When $\mathbf{y}$ is fixed in $\mathbb{C}^s$, $\nabla_{\mathbf{y}}$ induces a functional $\nabla_{\mathbf{y}}[\hat{\mathbf{x}}] : p \longrightarrow (\nabla_{\mathbf{y}} p)(\hat{\mathbf{x}})$. For any variable $\mathbf{u} = [u_1, \cdots, u_s]^\top$, the *gradient operator* $\Delta_{\mathbf{u}} \equiv \left[\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_s} \right]^\top$, whose "dot product" with a vector $\mathbf{v} = [v_1, \cdots, v_s]^\top$ is defined as

$$\mathbf{v} \cdot \Delta_{\mathbf{u}} \equiv v_1 \frac{\partial}{\partial u_1} + \cdots + v_s \frac{\partial}{\partial u_s}. \tag{31}$$

In particular, $\nabla_{\mathbf{y}} \equiv \mathbf{y} \cdot \Delta_{\mathbf{x}} \equiv \mathbf{y} \cdot \Delta_{\mathbf{x}_1}$ for any $\mathbf{y}$ of dimension s. Let $\mathbf{y}$ and $\mathbf{z}$ be auxiliary variables. Then, for any function $f(\mathbf{x})$,

$$\begin{array}{rcl} (\mathbf{y} \cdot \Delta_{\mathbf{x}_1})(\nabla_{\mathbf{z}} f(\mathbf{x}_1)) & = & \nabla_{\mathbf{y}} \nabla_{\mathbf{z}} f(\mathbf{x}_1), \qquad \mathbf{z} \cdot \Delta_{\mathbf{y}} f(\mathbf{x}_1) \equiv 0, \\ (\mathbf{z} \cdot \Delta_{\mathbf{y}})(\nabla_{\mathbf{y}} f(\mathbf{x}_1)) & = & (\mathbf{z} \cdot \Delta_{\mathbf{y}})(\mathbf{y} \cdot \Delta_{\mathbf{x}_1}) f(\mathbf{x}_1) = \nabla_{\mathbf{z}} f(\mathbf{x}_1). \end{array} \tag{32}$$

Let $\mathbf{f}_0(\mathbf{x}_1) \equiv \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \cdots, f_t(\mathbf{x})]^\top$ be a nonlinear system in variable vector $\mathbf{x}$ and $J_0(\mathbf{x})$ be its Jacobian matrix. Then

$$J_0(\mathbf{x})\,\mathbf{z} \;=\; \begin{bmatrix} \Delta_{\mathbf{x}} f_1(\mathbf{x})^\top \\ \vdots \\ \Delta_{\mathbf{x}} f_t(\mathbf{x})^\top \end{bmatrix} \mathbf{z} \;=\; \begin{bmatrix} \mathbf{z} \cdot \Delta_{\mathbf{x}} f_1(\mathbf{x}) \\ \vdots \\ \mathbf{z} \cdot \Delta_{\mathbf{x}} f_t(\mathbf{x}) \end{bmatrix} \;=\; \nabla_{\mathbf{z}} \mathbf{f}(\mathbf{x}_1).$$

The first depth-deflation step expands the system to $\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0}$ with

$$\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) \;\equiv\; \left[\begin{array}{c} \mathbf{f}_0(\mathbf{x}_1) \\ \begin{bmatrix} J_0(\mathbf{x}_1) \\ R_1 \end{bmatrix} \mathbf{x}_2 - \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{bmatrix} \end{array} \right] \;\equiv\; \begin{bmatrix} \mathbf{f}_0(\mathbf{x}_1) \\ \nabla_{\mathbf{x}_2} \mathbf{f}_0(\mathbf{x}_1) \\ R_1 \mathbf{x}_2 - \mathbf{e}_1 \end{bmatrix}. \tag{33}$$

The values of $\mathbf{x}_2 = \hat{\mathbf{x}}_2 \neq \mathbf{0}$ induce a functional $\nabla_{\hat{\mathbf{x}}_2}[\hat{\mathbf{x}}_1] \in \mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})$. If the zero $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ of $\mathbf{f}_1$ remains multiple, then the Jacobian $J_1(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ of $\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2)$ at $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ has a nullity $k_1 > 0$ and a nontrivial kernel. The depth-deflation process can be applied to $\mathbf{f}_1$ the same way as (33) applied to $\mathbf{f}_0$. Namely, we seek a zero $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3, \hat{\mathbf{x}}_4)$ to the system

$$\mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \;=\; \left[\begin{array}{c} \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) \\ \begin{bmatrix} J_1(\mathbf{x}_1, \mathbf{x}_2) \\ R_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{bmatrix} \end{array} \right]$$

where R_2 is any matrix of size $k_1 \times 2s$ that makes $\begin{bmatrix} J_1(\mathbf{x}_1, \mathbf{x}_2) \\ R_2 \end{bmatrix}$ full rank. By (30) – (32), equation $J_1\big(\mathbf{x}_1, \mathbf{x}_2\big) \begin{bmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix} = \mathbf{0}$ implies

$$\begin{array}{l} \left[\begin{array}{lcl} (\mathbf{x}_3 \cdot \Delta_{\mathbf{x}_1}) \mathbf{f}_0(\mathbf{x}_1) & + & (\mathbf{x}_4 \cdot \Delta_{\mathbf{x}_2}) \mathbf{f}_0(\mathbf{x}_1) \\ (\mathbf{x}_3 \cdot \Delta_{\mathbf{x}_1}) \nabla_{\mathbf{x}_2} \mathbf{f}_0(\mathbf{x}_1) & + & (\mathbf{x}_4 \cdot \Delta_{\mathbf{x}_2}) \nabla_{\mathbf{x}_2} \mathbf{f}_0(\mathbf{x}_1) \\ (\mathbf{x}_3 \cdot \Delta_{\mathbf{x}_1})(R_1 \mathbf{x}_2 - \mathbf{e}_1) & + & (\mathbf{x}_4 \cdot \Delta_{\mathbf{x}_2})(R_1 \mathbf{x}_2 - \mathbf{e}_1) \end{array} \right] \;=\; \left[\begin{array}{c} \nabla_{\mathbf{x}_3} \mathbf{f}_0(\mathbf{x}_1) \\ (\nabla_{\mathbf{x}_3} \nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_4}) \mathbf{f}_0(\mathbf{x}_1) \\ R_1 \mathbf{x}_4 \end{array} \right] \\ = \; \mathbf{0}. \end{array} \tag{34}$$

Thus, the second depth-deflation seeks a solution $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3, \hat{\mathbf{x}}_4)$ to equations

$$\mathbf{f}_0(\mathbf{x}_1) = \mathbf{0}, \quad \nabla_{\mathbf{x}_2} \mathbf{f}_0(\mathbf{x}_1) = \mathbf{0}, \quad \nabla_{\mathbf{x}_3} \mathbf{f}_0(\mathbf{x}_1) = \mathbf{0}, \quad (\nabla_{\mathbf{x}_3} \nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_4}) \mathbf{f}_0(\mathbf{x}_1) = \mathbf{0}. \tag{35}$$

It is important to note that $\hat{\mathbf{x}}_3 \neq \mathbf{0}$. Otherwise, from (34)

$$\begin{bmatrix} \nabla_{\hat{\mathbf{x}}_4} \mathbf{f}_0(\hat{\mathbf{x}}_1) \\ R_1 \hat{\mathbf{x}}_4 \end{bmatrix} \;\equiv\; \begin{bmatrix} J_0(\hat{\mathbf{x}}_1) \\ R_1 \end{bmatrix} \hat{\mathbf{x}}_4 \;=\; \mathbf{0},$$

which would lead to $\hat{\mathbf{x}}_4 = \mathbf{0}$, making it impossible for $R_2 \begin{bmatrix} \hat{\mathbf{x}}_3 \\ \hat{\mathbf{x}}_4 \end{bmatrix} = \mathbf{e}_1$.

After α depth-deflation steps, in general, we have an isolated zero $(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{2^\alpha})$ to the expanded system $\mathbf{f}_\alpha(\mathbf{x}_1, \cdots, \mathbf{x}_{2^\alpha})$, with Jacobian $J_\alpha(\mathbf{x}_1, \cdots, \mathbf{x}_{2^\alpha})$ of rank r_α. If $r_\alpha < 2^\alpha s$, then the next depth-deflation step seeks a zero to $\mathbf{f}_{\alpha+1}(\mathbf{x}_1, \cdots, \mathbf{x}_{2^{\alpha+1}}) = \mathbf{0}$ defined in (28).

Lemma 3 *Let* $\mathbf{f}_0(\mathbf{x}_1) \equiv \mathbf{f}(\mathbf{x})$ *be a system of* t *functions of* s *variables with a multiple zero* $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$. *Assume the depth-deflation process described above reaches the extended system* $\mathbf{f}_{\alpha+1}$ *in* (28) *with isolated zero* $(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{2^{\alpha+1}})$. *Then*

$$\hat{\mathbf{x}}_{2^j+1} \neq \mathbf{0}, \quad j = 0, 1, \cdots, \alpha.$$

Proof. The assertion is true for $j = 0$ and $j = 1$ as shown above. Let

$$\mathbf{y} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{2^{\alpha-1}} \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \mathbf{x}_{2^{\alpha-1}+1} \\ \vdots \\ \mathbf{x}_{2^{\alpha-1}+2^{\alpha-1}} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{x}_{2^{\alpha}+1} \\ \vdots \\ \mathbf{x}_{2^{\alpha}+2^{\alpha-1}} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{x}_{2^{\alpha}+2^{\alpha-1}+1} \\ \vdots \\ \mathbf{x}_{2^{\alpha}+2^{\alpha-1}+2^{\alpha-1}} \end{bmatrix}.$$

Then

$$J_\alpha(\mathbf{y}, \mathbf{z}) \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \cdot \Delta_{\mathbf{y}} \mathbf{f}_{\alpha\text{-}1}(\mathbf{y}) \\ [(\mathbf{u} \cdot \Delta_{\mathbf{y}})(\mathbf{z} \cdot \Delta_{\mathbf{y}}) + (\mathbf{v} \cdot \Delta_{\mathbf{y}})] \mathbf{f}_{\alpha\text{-}1}(\mathbf{y}) \\ R_{\alpha\text{-}1}\mathbf{v} \end{bmatrix} = \mathbf{0} \tag{36}$$

together with $\mathbf{u} = \mathbf{0}$ would imply

$$J_\alpha(\hat{\mathbf{y}}, \hat{\mathbf{z}}) \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ (\mathbf{v} \cdot \Delta_{\hat{\mathbf{y}}}) \mathbf{f}_{\alpha\text{-}1}(\hat{\mathbf{y}}) \\ R_{\alpha\text{-}1}\mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ J_{\alpha\text{-}1}(\hat{\mathbf{y}}) \\ R_{\alpha\text{-}1} \end{bmatrix} \mathbf{v} = \mathbf{0}$$

and thereby $\mathbf{v} = \mathbf{0}$ since $\begin{bmatrix} J_{\alpha\text{-}1}(\hat{\mathbf{y}}) \\ R_{\alpha\text{-}1} \end{bmatrix}$ is of full column rank. Therefore

$$\hat{\mathbf{u}} = \left(\hat{\mathbf{x}}_{2^\alpha+1}^\top, \cdots, \hat{\mathbf{x}}_{2^\alpha+2^{\alpha-1}}^\top \right)^\top \neq \mathbf{0}. \tag{37}$$

Moreover, from (36)

$$\mathbf{0} = \hat{\mathbf{u}} \cdot \Delta_{\mathbf{y}} \mathbf{f}_{\alpha\text{-}1}(\hat{\mathbf{y}}) \equiv J_{\alpha\text{-}1}(\hat{\mathbf{y}}) \hat{\mathbf{u}}. \tag{38}$$

It now suffices to show that for all η,

$$J_\eta(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{2^\eta}) \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_{2^\eta} \end{bmatrix} = \mathbf{0} \quad \text{and} \quad \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_{2^\eta} \end{bmatrix} \neq \mathbf{0} \tag{39}$$

would imply $\mathbf{w}_1 \neq \mathbf{0}$. Obviously, this is true for $\eta = 1$. Assume it is true for up to $\eta - 1$. Then, using the same argument for (37) and (38), we have (39) implying

$$\begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_{2^{\eta-1}} \end{bmatrix} \neq \mathbf{0} \quad \text{and} \quad J_{\eta-1} \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_{2^{\eta-1}} \end{bmatrix} = \mathbf{0}.$$

Thus $\mathbf{w}_1 \neq \mathbf{0}$ from the induction assumption. □

It is clear that the third depth-deflation, if necessary, adds variables $\mathbf{x}_5$, $\mathbf{x}_6$, $\mathbf{x}_7$, $\mathbf{x}_8$ and equations

$$\begin{array}{l} \nabla_{\mathbf{x}_5} \mathbf{f}(\mathbf{x}_1) = \mathbf{0}, \quad (\nabla_{\mathbf{x}_5} \nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_6}) \mathbf{f}(\mathbf{x}_1) = \mathbf{0}, \quad (\nabla_{\mathbf{x}_5} \nabla_{\mathbf{x}_3} + \nabla_{\mathbf{x}_7}) \mathbf{f}(\mathbf{x}_1) = \mathbf{0}, \\ (\nabla_{\mathbf{x}_5} \nabla_{\mathbf{x}_3} \nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_5} \nabla_{\mathbf{x}_4} + \nabla_{\mathbf{x}_3} \nabla_{\mathbf{x}_6} + \nabla_{\mathbf{x}_7} \nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_8}) \mathbf{f}(\mathbf{x}_1) = \mathbf{0}. \end{array} \tag{40}$$

Any solution $(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_8) \in \mathbb{C}^{8s}$ to (35) and (40) induces eight differential functionals

$$\begin{array}{l} 1, \quad \nabla_{\hat{\mathbf{x}}_2}, \quad \nabla_{\hat{\mathbf{x}}_3}, \quad \nabla_{\hat{\mathbf{x}}_5}, \\ \nabla_{\hat{\mathbf{x}}_3}\nabla_{\hat{\mathbf{x}}_2} + \nabla_{\hat{\mathbf{x}}_4}, \quad \nabla_{\hat{\mathbf{x}}_5}\nabla_{\hat{\mathbf{x}}_2} + \nabla_{\hat{\mathbf{x}}_6}, \quad \nabla_{\hat{\mathbf{x}}_5}\nabla_{\hat{\mathbf{x}}_3} + \nabla_{\hat{\mathbf{x}}_7}, \\ \nabla_{\hat{\mathbf{x}}_5}\nabla_{\hat{\mathbf{x}}_3}\nabla_{\hat{\mathbf{x}}_2} + \nabla_{\hat{\mathbf{x}}_5}\nabla_{\hat{\mathbf{x}}_4} + \nabla_{\hat{\mathbf{x}}_3}\nabla_{\hat{\mathbf{x}}_6} + \nabla_{\hat{\mathbf{x}}_7}\nabla_{\hat{\mathbf{x}}_2} + \nabla_{\hat{\mathbf{x}}_8} \end{array}$$

that vanish on $\mathbf{f}$ at $\hat{\mathbf{x}}_1$. In general, the α-th depth-deflation step produces a collection of 2^α differential functionals of order α or less that vanish on the system $\mathbf{f}$ at $\hat{\mathbf{x}}_1$. Also notice that the highest order differential terms are

$$\nabla_{\hat{\mathbf{x}}_2} \equiv \nabla_{\hat{\mathbf{x}}_{2^0+1}}, \quad \nabla_{\hat{\mathbf{x}}_3}\nabla_{\hat{\mathbf{x}}_2} \equiv \nabla_{\hat{\mathbf{x}}_{2^1+1}}\nabla_{\hat{\mathbf{x}}_{2^0+1}}, \quad \nabla_{\hat{\mathbf{x}}_5}\nabla_{\hat{\mathbf{x}}_3}\nabla_{\hat{\mathbf{x}}_2} \equiv \nabla_{\hat{\mathbf{x}}_{2^2+1}}\nabla_{\hat{\mathbf{x}}_{2^1+1}}\nabla_{\hat{\mathbf{x}}_{2^0+1}}$$

for depth-deflation steps 1, 2 and 3, respectively.

Actually these functionals induced by the depth-deflation method all belong to the dual space $\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})$. To show this, we define differential operators Φ_α, $\alpha = 1, 2, \cdots$ as follows.

$$\Phi_{\nu+1} = \sum_{\zeta=1}^{2^\nu} \mathbf{x}_{2^\nu+\zeta} \cdot \Delta_{\mathbf{x}_\zeta}, \quad \nu = 0, 1, \cdots . \tag{41}$$

Specifically,

$$\begin{array}{rcl} \Phi_1 & = & \mathbf{x}_2 \cdot \Delta_{\mathbf{x}_1} \\ \Phi_2 & = & \mathbf{x}_3 \cdot \Delta_{\mathbf{x}_1} + \mathbf{x}_4 \cdot \Delta_{\mathbf{x}_2} \\ \Phi_3 & = & \mathbf{x}_5 \cdot \Delta_{\mathbf{x}_1} + \mathbf{x}_6 \cdot \Delta_{\mathbf{x}_2} + \mathbf{x}_7 \cdot \Delta_{\mathbf{x}_3} + \mathbf{x}_8 \cdot \Delta_{\mathbf{x}_4}. \end{array}$$

For convenience, let Φ_0 represent the identity operator. Thus

$$\begin{array}{rcl} \Phi_0\mathbf{f}(\mathbf{x}_1) & = & \mathbf{f}(\mathbf{x}_1), \\ \Phi_1\mathbf{f}(\mathbf{x}_1) & = & \nabla_{\mathbf{x}_2}\mathbf{f}(\mathbf{x}_1), \\ \Phi_2\mathbf{f}(\mathbf{x}_1) & = & \nabla_{\mathbf{x}_3}\mathbf{f}(\mathbf{x}_1), \\ \Phi_2 \circ \Phi_1\mathbf{f}(\mathbf{x}_1) & = & (\mathbf{x}_3 \cdot \Delta_{\mathbf{x}_1})\nabla_{\mathbf{x}_2}\mathbf{f}(\mathbf{x}_1) + (\mathbf{x}_4 \cdot \Delta_{\mathbf{x}_2})\nabla_{\mathbf{x}_2}\mathbf{f}(\mathbf{x}_1) \quad = \quad (\nabla_{\mathbf{x}_3}\nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_4})\mathbf{f}(\mathbf{x}_1) \end{array}$$

etc. For any expanded system $\mathbf{f}_\alpha(\mathbf{x}_1, \cdots, \mathbf{x}_{2^\alpha})$ generated in the depth-deflation process, its Jacobian $J_\alpha(\mathbf{x}_1, \cdots, \mathbf{x}_{2^\alpha})$ satisfies

$$J_\alpha(\mathbf{x}_1, \cdots, \mathbf{x}_{2^\alpha}) \begin{bmatrix} \mathbf{x}_{2^\alpha+1} \\ \vdots \\ \mathbf{x}_{2^\alpha+2^\alpha} \end{bmatrix} = \Phi_{\alpha+1}\mathbf{f}_\alpha(\mathbf{x}_1, \cdots, \mathbf{x}_{2^\alpha}).$$

It is easy to see that (35) and (40) can be written as

$$\begin{array}{rcl} \Phi_0\mathbf{f}(\mathbf{x}_1) & = & \mathbf{0}, \\ \Phi_1\mathbf{f}(\mathbf{x}_1) & = & \mathbf{0}, \\ \Phi_2\mathbf{f}(\mathbf{x}_1) & = & \mathbf{0}, \quad \Phi_2 \circ \Phi_1\mathbf{f}(\mathbf{x}_1) = \mathbf{0}, \\ \Phi_3\mathbf{f}(\mathbf{x}_1) & = & \mathbf{0}, \quad \Phi_3 \circ \Phi_1\mathbf{f}(\mathbf{x}_1) = \mathbf{0}, \quad \Phi_3 \circ \Phi_2\mathbf{f}(\mathbf{x}_1) = \mathbf{0}, \quad \Phi_3 \circ \Phi_2 \circ \Phi_1\mathbf{f}(\mathbf{x}_1) = \mathbf{0}. \end{array}$$

As a consequence, Theorem 4 given in §3 provides an upper bound, *the depth*, on the number of depth-deflation steps required to regularize the singularity at the multiple zero. This bound substantially improves the result in [11, Theorem 3.1]. In fact, our version of the deflation method deflates *depth* rather than the multiplicity as suggested in [11].

We now give a proof of Theorem 4.

Proof of Theorem 4. We first claim that the α-th depth-deflation step that solves $\mathbf{f}(\mathbf{x}_1) = \mathbf{0}$ induces *all* differential functionals in the form of $\Phi_{\mu_1} \circ \cdots \circ \Phi_{\mu_k} \mathbf{f}(\mathbf{x}_1) = \mathbf{0}$ at $(\mathbf{x}_1, \cdots, \mathbf{x}_{2^\alpha}) = (\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{2^\alpha})$ with $\alpha \geq \mu_1 > \mu_2 > \cdots > \mu_k > 0$ and $1 \leq k \leq \alpha$. This is clearly true for $\alpha = 1$ since $\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0}$ induces

$$\Phi_0 \mathbf{f}(\mathbf{x}_1) \quad = \quad \Phi_1 \mathbf{f}(\mathbf{x}_1) \quad = \quad \mathbf{0}.$$

Assume the claim is true for $\alpha - 1$. For α-th depth-deflation, consider a functional in the following form

$$\Phi_{\mu_1} \circ \cdots \circ \Phi_{\mu_k} \mathbf{f}(\mathbf{x}_1) \quad = \quad \mathbf{0}, \quad \alpha \geq \mu_1 > \mu_2 > \cdots > \mu_k \geq 0. \tag{42}$$

If $\mu_1 < \alpha$, then such a functional must already be induced from solving $\mathbf{f}_{\alpha-1} = \mathbf{0}$. On the other hand, if $\mu_1 = \alpha$, then

$$\Phi_{\mu_2} \circ \cdots \circ \Phi_{\mu_k} \mathbf{f}(\mathbf{x}_1) \quad = \quad \mathbf{0}, \quad \alpha - 1 \geq \mu_2 > \cdots > \mu_k \geq 0$$

is in $\mathbf{f}_{\alpha-1} = \mathbf{0}$. Therefore $\Phi_\alpha F_{\alpha-1}$ induces the functional in (42).

Next, the functional in (42) satisfies closedness condition (11). To show this, let p be any polynomial in variables $\mathbf{x}$. By applying the product rule $\Phi_\alpha(f\, g) = (\Phi_\alpha\, f)\, g + (\Phi_\alpha\, g)\, f$ in an induction,

$$\Phi_{\mu_1} \circ \cdots \circ \Phi_{\mu_k}(p f_i) = \sum_{\{\eta_1, \cdots, \eta_j\} \subset \{\mu_1, \cdots, \mu_k\}} p_{\eta_1 \cdots \eta_j} \Phi_{\eta_1} \circ \cdots \circ \Phi_{\eta_j} f_i$$

where $\eta_1 > \cdots > \eta_j$ and $p_{\eta_1 \cdots \eta_j}$ is a polynomial generated by applying Φ_j's on p. Therefore $\Phi_{\mu_1} \circ \cdots \circ \Phi_{\mu_k}(p f_i) = 0$ at $(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{2^\alpha})$ since $\Phi_{\eta_1} \circ \cdots \circ \Phi_{\eta_j} f_i = 0$, showing that functionals in the form of (42) are all in the dual space $\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})$.

Finally, the highest order part of the differential functional $\Phi_\alpha \circ \Phi_{\alpha-1} \circ \cdots \circ \Phi_1$ is

$$\prod_{j=0}^{\alpha-1} (\hat{\mathbf{x}}_{2^j+1} \cdot \Delta_{\mathbf{x}}) \quad \equiv \quad \prod_{j=0}^{\alpha-1} \nabla_{\hat{\mathbf{x}}_{2^j+1}}$$

which is of order α since $\hat{\mathbf{x}}_{2^j+1} \neq \mathbf{0}$ by Lemma 3. However, differential orders of all functionals in $\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})$ are bounded by $\delta_{\hat{\mathbf{x}}}(\mathbf{f})$, so is α. □

In general, Theorem 4 does not guarantee those 2^k functionals are linearly independent.

From computing experiments, the number k of depth-deflation steps also correlates to the breadth $\beta_{\hat{\mathbf{x}}}(\mathbf{f})$. Especially when $\beta_{\hat{\mathbf{x}}}(\mathbf{f}) = 1$, it appears that k always reaches its maximum. This motivates the special case breadth-one algorithm which will be presented in §3.3. On the other hand, when breadth $\beta_{\hat{\mathbf{x}}}(\mathbf{f}) > 1$, very frequently the depth-deflation process pleasantly terminates only after *one* depth-deflation step regardless of the depth or multiplicity. A possible explanation for such a phenomenon is as follows. At each depth-deflation step, say the first, the isolated zero $\hat{\mathbf{z}}$ to the system (33) is multiple only if there is a differential functional in the form of $\nabla_{\mathbf{x}_3} \nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_4}$ in $\mathcal{D}^2_{\hat{\mathbf{x}}}(\mathbf{f})$ while $R_1 \mathbf{x}_2 = \mathbf{e}_1$ and $R_1 \mathbf{x}_4 = \mathbf{0}$ for a randomly chosen R_1. In most of the polynomial systems we have tested, functionals in this special form *rarely* exist in $\mathcal{D}^2_{\hat{\mathbf{x}}}(\mathbf{f})$ when $\beta_{\hat{\mathbf{x}}}(\mathbf{f}) > 1$. If no such functionals exist in $\mathcal{D}^2_{\hat{\mathbf{x}}}(\mathbf{f})$, the zero $\hat{\mathbf{z}}$ must be a simple zero of $\tilde{F}$ in (33) according to Theorem 4, therefore the depth-deflation ends at $k = 1$ step.

3.3 Special case: dual space of breadth one

Consider a nonlinear system $\mathbf{f} = [f_1, \cdots, f_t]^\top$ having breadth-one at an isolated zero $\hat{\mathbf{x}}$, namely $\beta_{\hat{\mathbf{x}}}(\mathbf{f}) = 1$. The Hilbert function is $\{1, 1, \cdots, 1, 0, \cdots\}$, making the depth $\delta_{\hat{\mathbf{x}}}(\mathbf{f})$ one less than the multiplicity $\mathit{dim}\big(\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})\big)$. This special case includes the most fundamental univariate equation $f(x) = 0$ at a multiple zero.

As mentioned above, the general depth-deflation method derived in §3.1 *always* exhausts the maximal number of steps in this case, and the final system is expanded undesirably from $t \times s$ to over $(2^{m-1}t) \times (2^{m-1}s)$ at an m-fold zero. To overcome this exponential growth of the system size, we shall modify the depth-deflation process for breadth-one system in this section in which the regularized system is of the size close to $(mt) \times (ms)$, and upon solving the system, a complete basis for the dual space $\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})$ is obtained as a by-product.

Denote $\mathbf{x} = \mathbf{x}_1$ and the zero $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1$ as in §3.1. It follows from (20), the breadth $\beta_{\hat{\mathbf{x}}}(\mathbf{f}) = \mathrm{h}(1) = \mathit{nullity}\,(\,J_0(\hat{\mathbf{x}}_1)\,) = 1$ implies system (33), simplifying to $\begin{bmatrix} J_0(\hat{\mathbf{x}}_1) \\ \mathbf{b}^{\mathsf{H}} \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$ in the variable vector $\mathbf{x}_2$, has a unique solution $\hat{\mathbf{x}}_2 \in \mathbb{C}^s$ for randomly chosen vector $\mathbf{b} \in \mathbb{C}^s$. Similar to the general depth-deflation method in § 3.1, the first step of depth-deflation is to set up an expanded system:

$$\mathbf{g}_1\,(\mathbf{x}_1, \mathbf{x}_2) = \begin{bmatrix} \mathbf{h}_0(\mathbf{x}_1) \\ \mathbf{h}_1(\mathbf{x}_1, \mathbf{x}_2) \end{bmatrix} \tag{43}$$

$$\text{where } \mathbf{h}_0(\mathbf{x}_1) \equiv \mathbf{f}(\mathbf{x}) \text{ and } \mathbf{h}_1(\mathbf{x}_1, \mathbf{x}_2) = \begin{bmatrix} J_0(\mathbf{x}_1)\,\mathbf{x}_2 \\ \mathbf{b}^{\mathsf{H}}\mathbf{x}_2 - 1 \end{bmatrix} \equiv \begin{bmatrix} \nabla_{\mathbf{x}_2}\mathbf{f}(\mathbf{x}_1) \\ \mathbf{b}^{\mathsf{H}}\mathbf{x}_2 - 1 \end{bmatrix}.$$

The system $\mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2)$ has an isolated zero $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$. If the Jacobian $J_1(\mathbf{x}_1, \mathbf{x}_2)$ of $g_1(\mathbf{x}_1, \mathbf{x}_2)$ is of full rank at $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$, then the system is regularized and the depth-deflation process terminates. Otherwise, there is a nonzero vector $(\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{C}^{2s}$ such that

$$J_1(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \equiv \begin{bmatrix} \nabla_{\mathbf{v}_1}\mathbf{f}(\hat{\mathbf{x}}_1) \\ (\nabla_{\mathbf{v}_1}\nabla_{\hat{\mathbf{x}}_2} + \nabla_{\mathbf{v}_2})\mathbf{f}(\hat{\mathbf{x}}_1) \\ \mathbf{b}^{\mathsf{H}}\mathbf{v}_2 \end{bmatrix} = \mathbf{0}. \tag{44}$$

Since the Jacobian $J_0(\hat{\mathbf{x}})$ of $\mathbf{f}$ at $\hat{\mathbf{x}}_1$ is of nullity one, there is a constant $\gamma \in \mathbb{C}$ such that $\mathbf{v}_1 = \gamma\hat{\mathbf{x}}_2$. Equation (44) together with $\beta_{\hat{\mathbf{x}}_0}(\mathbf{f}) = 1$ and $(\mathbf{v}_1, \mathbf{v}_2) \neq (\mathbf{0}, \mathbf{0})$ imply $\gamma \neq 0$. Consequently we may choose $\gamma = 1$, namely $\mathbf{v}_1 = \hat{\mathbf{x}}_2$. Setting $\hat{\mathbf{x}}_3 = \mathbf{v}_2$, the system

$$\mathbf{g}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \equiv \begin{bmatrix} \mathbf{h}_0(\mathbf{x}_1) \\ \mathbf{h}_1(\mathbf{x}_1, \mathbf{x}_2) \\ \mathbf{h}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{x}_1) \\ \nabla_{\mathbf{x}_2}\mathbf{f}(\mathbf{x}_1) \\ \mathbf{b}^{\mathsf{H}}\mathbf{x}_2 - 1 \\ (\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_3})\mathbf{f}(\mathbf{x}_1) \\ \mathbf{b}^{\mathsf{H}}\mathbf{x}_3 \end{bmatrix} \tag{45}$$

$$\text{where } \mathbf{h}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{bmatrix} (\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_2} + \nabla_{\mathbf{x}_3})\mathbf{f}(\mathbf{x}_1) \\ \mathbf{b}^{\mathsf{H}}\mathbf{x}_3 \end{bmatrix}$$

has an isolated zero $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3)$. In general, if an isolated zero $(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{\gamma+1})$ to the system

$$\mathbf{g}_\gamma(\mathbf{x}_1, \cdots, \mathbf{x}_{\gamma+1}) = \begin{bmatrix} \mathbf{h}_0(\mathbf{x}_1) \\ \mathbf{h}_1(\mathbf{x}_1, \mathbf{x}_2) \\ \vdots \\ \mathbf{h}_\gamma(\mathbf{x}_1, \cdots, \mathbf{x}_{\gamma+1}) \end{bmatrix}$$

remains singular, or the Jacobian $J_\gamma(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{\gamma+1})$ is rank-deficient, then there is a non-zero solution to the homogeneous system

$$J_\gamma(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{\gamma+1}) \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{\gamma+1} \end{bmatrix} \equiv \begin{bmatrix} J_{\gamma-1}(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_\gamma) \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_\gamma \end{bmatrix} \\ * \end{bmatrix} = \mathbf{0}.$$

Therefore, by setting $\mathbf{u}_1 = \hat{\mathbf{x}}_2$, $\mathbf{u}_2 = \hat{\mathbf{x}}_3$, $\cdots$, $\mathbf{u}_\gamma = \hat{\mathbf{x}}_{\gamma+1}$, we take its unique solution $\mathbf{u}_{\gamma+1}$ as $\hat{\mathbf{x}}_{\gamma+2}$.

The pattern of this depth-deflation process can be illustrated by defining

$$\Psi = \sum_{\eta=1}^{\infty} \mathbf{x}_{\eta+1} \cdot \Delta_{\mathbf{x}_\eta}. \tag{46}$$

When applying Ψ to any function f in (vector) variables, say $\mathbf{x}_1, \cdots, \mathbf{x}_\sigma$, the resulting Ψf is a *finite* sum since $\Delta_{\mathbf{x}_\mu} f = \mathbf{0}$ for $\mu \geq \sigma + 1$. Thus,

$$\mathbf{h}_1(\mathbf{x}_1, \mathbf{x}_2) = \begin{bmatrix} \Psi \mathbf{h}_0(\mathbf{x}_1) \\ \mathbf{b}^{\mathsf{H}} \mathbf{x}_2 - 1 \end{bmatrix}, \quad \mathbf{h}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{bmatrix} \Psi \mathbf{h}_1(\mathbf{x}_1, \mathbf{x}_2) \\ \mathbf{b}^{\mathsf{H}} \mathbf{x}_3 - 1 \end{bmatrix} \quad \text{and}$$

$$\mathbf{h}_\nu(\mathbf{x}_1, \cdots, \mathbf{x}_\nu) = \begin{bmatrix} \overbrace{\Psi \circ \Psi \circ \cdots \circ \Psi}^{\nu-1} \mathbf{h}_1(\mathbf{x}_1, \mathbf{x}_2), \\ \\ \mathbf{b}^{\mathsf{H}} \mathbf{x}_{\nu+1} \end{bmatrix}, \quad \text{for} \quad \nu \geq 2. \tag{47}$$

For instance, with $\mathbf{h}_1$ and $\mathbf{h}_2$ in (43) and (45) respectively, we have

$$\mathbf{h}_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \begin{bmatrix} (\nabla_{\mathbf{x}_2} \nabla_{\mathbf{x}_2} \nabla_{\mathbf{x}_2} + 3 \nabla_{\mathbf{x}_2} \nabla_{\mathbf{x}_3} + \nabla_{\mathbf{x}_4}) \mathbf{h}_0(\mathbf{x}_1) \\ \mathbf{b}^{\mathsf{H}} \mathbf{x}_4 \end{bmatrix}.$$

If, say, $\mathbf{h}_3 = \mathbf{0}$ at $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3, \hat{\mathbf{x}}_4)$, a functional $f \longrightarrow (\nabla_{\hat{\mathbf{x}}_2} \nabla_{\hat{\mathbf{x}}_2} \nabla_{\hat{\mathbf{x}}_2} + 3 \nabla_{\hat{\mathbf{x}}_2} \nabla_{\hat{\mathbf{x}}_3} + \nabla_{\hat{\mathbf{x}}_4}) f(\mathbf{x}_1)$ is obtained and it vanishes on the system $\mathbf{f}$. The original system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ provides a trivial functional $\partial_{0 \cdots 0} : f \to f(\hat{\mathbf{x}}_1)$. By the following lemma those functionals are all in the dual space.

Lemma 4 *Let $\mathbf{f} = [f_1, \cdots, f_t]^\top$ be a nonlinear system with an isolated zero $\hat{\mathbf{x}} \in \mathbb{C}^s$. Write $\mathbf{g}_0 = \mathbf{f}$, $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$ and $\mathbf{x}_1 = \mathbf{x}$. For any $\gamma \in \{1, 2, \cdots\}$, let $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \cdots, \hat{\mathbf{x}}_{\gamma+1})$ be a zero of*

$$\mathbf{g}_\gamma(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{\gamma+1}) = \begin{bmatrix} \mathbf{h}_0(\mathbf{x}_1) & \\ \vdots & \ddots \\ \mathbf{h}_\gamma(\mathbf{x}_1, \cdots, \mathbf{x}_{\gamma+1}) & \end{bmatrix}. \tag{48}$$

Then the functionals derived from $\mathbf{g}_\gamma(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{\gamma+1}) = \mathbf{0}$ constitutes a linearly independent subset of the dual space $\mathcal{D}_{\hat{\mathbf{x}}_0}(\mathbf{f})$.

Proof. By a rearrangement if necessary, finding a zero to $\mathbf{g}_\gamma(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{\gamma+1})$ is equivalent to solving

$$\begin{aligned} \mathbf{f}(\mathbf{x}_1) &= \mathbf{0}, & \mathbf{b}^{\mathsf{H}} \mathbf{x}_2 &= 1, \\ \Psi \mathbf{f}(\mathbf{x}_1) &= \mathbf{0}, & \mathbf{b}^{\mathsf{H}} \mathbf{x}_3 &= 0, \\ &\vdots & &\vdots \\ \Psi \circ \cdots \circ \Psi \mathbf{f}(\mathbf{x}_1) &= \mathbf{0}, & \mathbf{b}^{\mathsf{H}} \mathbf{x}_{\gamma+1} &= 0. \end{aligned} \tag{49}$$

for $(\mathbf{x}_1, \cdots, \mathbf{x}_{\gamma+1}) \in \mathbb{C}^{(\gamma+1)s}$. Let $(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{\gamma+1})$ be an isolated zero. Then each $\Psi \circ \cdots \circ \Psi$ induces a differential functional

$$\rho_\alpha \quad : \quad f \longrightarrow \overbrace{\Psi \circ \cdots \circ \Psi}^{\alpha} f \Big|_{(\mathbf{x}_1, \cdots, \mathbf{x}_{\alpha+1}) = (\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{\alpha+1})}, \quad \text{for} \quad \alpha = 0, 1, \cdots, \gamma. \tag{50}$$

Those functionals vanish on $f_1, \cdots, f_t$ because of (49). Since Ψ satisfies product rule $\Psi(fg) = (\Psi f)g + f(\Psi g)$ for any functions f and g in finitely many variables among $\mathbf{x}_1, \mathbf{x}_2, \cdots$, for any polynomial $p \in \mathbb{C}[\mathbf{x}_1]$, we have, for $\alpha = 0, 1, \cdots, \gamma$ and $i = 1, \cdots, t$,

$$\rho_\alpha(p f_i) \quad = \quad \sum_{j=0}^{\alpha} \binom{\alpha}{j} (\overbrace{\Psi \circ \cdots \circ \Psi}^{j} p)(\overbrace{\Psi \circ \cdots \circ \Psi}^{\alpha - j} f_i) \Big|_{(\mathbf{x}_1, \cdots, \mathbf{x}_{\alpha+1}) = (\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{\alpha+1})} = 0.$$

Namely, ρ_α's satisfy the closedness condition (11), so they belong to $\mathcal{D}_{\hat{\mathbf{x}}_1}(\mathbf{f})$.

The leading (i.e., the highest order differential) term of ρ_α is $\overbrace{\nabla_{\hat{\mathbf{x}}_2} \cdots \nabla_{\hat{\mathbf{x}}_2}}^{\alpha}$ which is of order α since $\hat{\mathbf{x}}_2 \neq \mathbf{0}$. Therefore, they are linearly independent. □

Theorem 5 (Breadth-one Deflation Theorem) *Let $\hat{\mathbf{x}}$ be an isolated multiple zero of the nonlinear system $\mathbf{f} = [f_1, \cdots, f_t]^\top$ with breadth $\beta_{\hat{\mathbf{x}}}(\mathbf{f}) = 1$. Then there is an integer $\gamma \leq \delta_{\hat{\mathbf{x}}}(\mathbf{f})$ such that, for almost all $\mathbf{b} \in \mathbb{C}^s$, the system $\mathbf{g}_\gamma$ in (48) has a simple zero $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \cdots, \hat{\mathbf{x}}_{\gamma+1})$ which induces $\gamma+1$ linearly independent functionals in $\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f})$.*

Proof. A straightforward consequence of Lemma 4. □

While the general depth-deflation method usually terminates with one or two steps of system expansion for systems of breadth higher than one, the breadth one depth-deflation *always* terminates at step $\gamma = \delta_{\hat{\mathbf{x}}}(\mathbf{f})$ exactly. Summarizing the above elaboration, we give the pseudo-code of an efficient algorithm for computing the multiplicity structure of the breadth one case as follows:

Algorithm BreadthOneMultiplicity

`Input:` Nonlinear system $\mathbf{f} = [f_1, \ldots, f_t]^{\mathsf{H}}$, zero $\hat{\mathbf{x}}_1 \in \mathbb{C}^s$

- `set` random vectors $\mathbf{b} \in \mathbb{C}^s$ and `obtain` $\hat{\mathbf{x}}_2$ by solving $\begin{bmatrix} J(\hat{\mathbf{x}}_1) \\ \mathbf{b}^{\mathsf{H}} \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$
- `initialize` $\mathbf{p}_2(\mathbf{x}_1, \mathbf{x}_2) = J(\mathbf{x}_1)\mathbf{x}_2$
- `for` $k = 2, 3, \ldots$ `do`
  - `set` $\mathbf{d}_k(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \sum_{j=1}^{k-1} \hat{\mathbf{x}}_{j+1} \cdot \Delta_{\mathbf{x}_j} \mathbf{p}_k(\mathbf{x}_1, \ldots, \mathbf{x}_k)$
  - `solve` for $\mathbf{x}_{k+1} = \hat{\mathbf{x}}_{k+1}$ in the system

$$\begin{bmatrix} J(\hat{\mathbf{x}}_1) \\ \mathbf{b}^{\mathsf{H}} \end{bmatrix} \mathbf{x}_{k+1} = \begin{bmatrix} \mathbf{d}_k(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_k) \\ 0 \end{bmatrix} \tag{51}$$

  - `if` the equation (51) has no solution, `set` $\gamma = k - 1$ and `break` the loop; otherwise, `set`

$$\mathbf{p}_{k+1}(\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}) = \Psi\, \mathbf{p}_k(\mathbf{x}_1, \ldots, \mathbf{x}_k) \equiv \mathbf{d}_k(\mathbf{x}_1, \ldots, \mathbf{x}_k) + J(\mathbf{x}_1)\mathbf{x}_{k+1}$$

```
    end do
```
`Output:` multiplicity $\gamma+1$ and functionals $\rho_0,\ \rho_1,\ \ldots,\ \rho_\gamma$ as in (50)

Example 6 Consider the sequence of nonlinear systems

$$\mathbf{f}_k(x,y,z) \quad = \quad [\,x^2\,\sinh y,\ y-z^2,\ z+\sin x^k\,] \tag{52}$$

of breadth one at the isolated zero $(0,0,0)$. The multiplicities $dim\big(\mathcal{D}_{(0,0,0)}(\mathbf{f}_k)\big) = 2(k+1)$ and depth $\delta_{(0,0,0)}(\mathbf{f}_k) = 2k+1$. The computing time for the general algorithm NonlinearSystemMultiplicity grows substantially when the depth increases. As shown in Table 2, Algorithm BreadthOneMultiplicity is much faster in this special case. □

k:	1	2	3	4	5
depth $\delta_{(0,0,0)}(\mathbf{f}_k)$:	3	5	7	9	11
NonlinearSystemMultiplicity	1.86	19.91	152.17	993.97	4829.55
BreadthOneMultiplicity	0.25	0.64	1.09	2.06	3.93

Table 2: Execution time (seconds) of two algorithms for computing the multiplicities of the breadth one systems $\mathbf{f}_k,\ k=1,2,\ldots,5$ in (52).

In our extensive computing experiments, Algorithm BreadthOneMultiplicity always produces a complete dual basis $\rho_0,\ldots,\rho_\gamma$ without premature termination. We believe the following conjecture is true.

Conjecture 1 *Under the assumptions of Theorem 5, Algorithm* BreadthOneMultiplicity *terminates at* $\gamma=\delta_{\hat{\mathbf{x}}}(\mathbf{f})$ *and generates a complete basis for the dual space*

$$\mathcal{D}_{\hat{\mathbf{x}}}(\mathbf{f}) \quad = \quad span\{\rho_0,\rho_1,\ldots,\rho_\gamma\}.$$

4 Algebraic foundation of the multiplicity

In previous sections, we elaborated the duality formulation of the multiplicity and its identification as well as the computation of multiple zeros without using terminology and theory of algebraic geometry. We postponed the proofs of the Local Finiteness Theorem (Theorem 1), the Multiplicity Consistency Theorem (Theorem 2) and the Perturbation Invariance Theorem (Theorem 3) until after a more solid theoretical foundation for the multiplicity is established in this section. For general readership, we shall attempt to make this section self contained with a prerequisite of a standard first year graduate course in abstract algebra. Perhaps an exception is the proofs in §4.1 which require a familiarity with the basic theory of functions of several complex variables as outlined in [23, Chapters 3,4].

4.1 Analytic Preliminaries

Let $\mathbb{C}[x_1,\ldots,x_s]$ be the ring of polynomials in the variables $x_1,\ldots,x_s$ and let $\mathbb{C}\{x_1,\ldots,x_s\}$ denote the ring of convergent power series centered at $\mathbf{0}$, that is, each element $f\in\mathbb{C}\{x_1,\ldots,x_s\}$

converges in some open set $\mathcal{V}_f$ about $\mathbf{0}$, see [4, 23]. For a set $F = \{f_1, \ldots, f_t\}$ of analytic functions and a ring $\mathcal{R}$ of functions on open set $\mathcal{U}$ and assume $\mathbf{0} \in \mathcal{U}$, let $F\mathcal{R}$ denote the ideal

$$F\mathcal{R} \;=\; \big\{ f_1 g_1 + \cdots + f_t g_t \;\big|\; g_1, \ldots, g_t \in \mathcal{R} \big\}. \tag{53}$$

For an analytic function f, define $\mathit{ord}(f) = m$ to be the smallest integer such that a term of the multivariate Taylor series of f at $\mathbf{0}$ of total degree m has a nonzero coefficient. Define

$$\mathit{jet}(f,k) \;=\; \sum_{|\alpha| \le k} f_\alpha \tag{54}$$

where for $\alpha = x_1^{j_1} \ldots x_s^{j_s}$, $|\alpha| = j_1 + \cdots + j_s$ is the total degree and f_α is the term $c_\alpha \alpha$ in a Taylor series expansion of f about $\mathbf{0}$. Note that for a fixed k, the map $\mathit{jet}(\cdot, k)$ is linear on $\mathcal{R}$ and $\mathit{ord}(f - \mathit{jet}(f,m)) > m$. An *open polydisc* in $\mathbb{C}^s$ at a point $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$ is defined as

$$\Delta(\hat{\mathbf{x}}, \mathbf{r}) \;=\; \{ \mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{C}^s \;\big|\; |a_i - \hat{x}_i| < r_i,\ i = 1, \ldots, s \}$$

for an array $\mathbf{r} = [r_1, \ldots, r_s] \in \mathbb{R}_+^s$ of positive real numbers. From an analytic point of view an *isolated zero* is defined by

Definition 2 *A point* $\hat{\mathbf{x}}$ *is an* isolated zero *of a system* $F = \{f_1, \ldots, f_t\}$ *if there is an open polydisc* $\Delta(\hat{\mathbf{x}}, \mathbf{r})$ *in which* $\hat{\mathbf{x}}$ *is the only zero of* F.

Lemma 5 *Let* $\mathcal{R}$ *be the ring of analytic functions on open set* $\mathcal{U} \subseteq \mathbb{C}^s$ *and assume* $\hat{\mathbf{x}} = \mathbf{0} \in \mathcal{U}$. *Let* $F = \{f_1, \ldots, f_t\} \subset \mathcal{R}$ *be a system of analytic functions with common zero* $\hat{\mathbf{x}}$. *Then the following are equivalent:*

(i) *The point* $\hat{\mathbf{x}} = \mathbf{0} \in \mathcal{U}$ *is an isolated zero of* F.
(ii) *For each* $j \in \{1, \ldots, s\}$ *there is a positive integer* e_j *such that* $x_j^{e_j} \in F\mathbb{C}\{x_1, \ldots, x_s\}$.
(iii) *For each* $j \in \{1, \ldots, s\}$ *there is a positive integer* e_j *and an analytic function* $p_j \in \mathcal{R}$ *with* $\mathit{ord}(p_j) > e_j$ *and* $x_j^{e_j} + p_j \in F\mathbb{C}[x_1, \ldots, x_s]$.

Proof. We first prove (i) implies (ii) following Rükert's Nullstellensatz [23, Theorem 4.5.5]. Let $F^* = F\mathbb{C}\{x_1, \ldots, x_s\}$ be the ideal generated by F in $\mathbb{C}\{x_1, \ldots, x_s\}$. Then loc F^*, the locus of F^*, is the germ of the one element variety $\{\mathbf{0}\}$, using the notation in [23]. By the Nullstellensatz, the radical of $F^* = \sqrt{F^*}$ is the maximal ideal of $\mathbb{C}\{x_1, \ldots, x_s\}$ at $\mathbf{0}$. Notice that

$$\sqrt{F^*} \;=\; \{\, f \in \mathbb{C}\{x_1, \ldots, x_s\} \;\big|\; f^k \in F^* \quad \text{for some integer } k > 0 \,\}.$$

However, monomials $x_1, \ldots, x_s$ all belong to this maximal ideal. As a result, there exist positive integers $e_1, \ldots, e_s$ such that $x_1^{e_1}, \ldots, x_s^{e_s} \in F^*$, proving assertion (ii).

To prove (ii) implies (iii), let $j \in \{1, \ldots, s\}$. The assertion (ii) yields the existence of $g_1, \ldots, g_t \in \mathbb{C}\{x_1, \ldots, x_s\}$ with

$$x_j^{e_j} \;=\; g_1 f_1 + \cdots + g_t f_t. \tag{55}$$

Let $\tilde{g}_i = g_i - \mathit{jet}(g_i, e_j)$. Then $\mathit{ord}(\tilde{g}_i) > e_j$. Therefore, for each j, from (55)

$$\begin{aligned} x_j^{e_j} &= (\mathit{jet}(g_1, e_j) + \tilde{g}_1) f_1 + \cdots + (\mathit{jet}(g_t, e_j) + \tilde{g}_t) f_t \\ &= \big(\mathit{jet}(g_1, e_j) f_1 + \cdots + \mathit{jet}(g_t, e_j) f_t\big) + \big(\tilde{g}_1 f_1 + \cdots + \tilde{g}_t f_t\big) \end{aligned} \tag{56}$$

and

$$x_j^{e_j} + p_j \;\; = \;\; \mathit{jet}(g_1, e_j)f_1 + \cdots + \mathit{jet}(g_t, e_j)f_t \;\; \in \;\; F\mathbb{C}[x_1, \ldots, x_s] \tag{57}$$

where $p_j = -\big(\tilde{g}_1 f_1 + \cdots + \tilde{g}_t f_t\big)$. Consequently $\mathit{ord}(p_j) > e_j$ and the right hand side of (57) is in $F\mathcal{R}$ since $\mathit{jet}(g_i, e_j)$'s are polynomials and thus belong to $\mathcal{R}$ for all $i \in \{1, \ldots, t\}$. Moreover, since $x_j^{e_j}$ is also in $\mathcal{R}$, it follows that actually p_j is in $\mathcal{R}$.

For (iii) implying (i), assume the assertion (iii) holds. Then by Schwarz's Lemma [23, Exercise 4, p. 35] for each $j = 1, \ldots, s$ there exists a constant K_j such that $|p_j(\mathbf{x})| \leq K_j \|\mathbf{x}\|^{e_j+1}$ in a polydisc $\Delta(\mathbf{0}, [t_j, \ldots, t_j])$ of $\mathbf{0}$. Let $r_j = \min\{t_j, \frac{1}{2K_j}\}$ for $j = 1, \ldots, s$. With $\mathbf{r} = [r_1, \ldots, r_s]$, let

$$\mathcal{V}_j \;\; = \;\; \big\{\, \mathbf{x} = (x_1, \ldots, x_s) \in \Delta(\mathbf{0}, \mathbf{r}) \setminus \{\mathbf{0}\} \;\big|\; \max_{1 \leq i \leq s} |x_i| = |x_j| \,\big\}$$

for $j = 1, \ldots, s$. Now in each $\mathcal{V}_j$, we have

$$|p_j(\mathbf{x})| \;\; \leq \;\; K_j \|\mathbf{x}\|^{e_j+1} \;\; \leq \;\; K_j |x_j|^{e_j+1} \;\; < \;\; K_j r_j |x_j|^{e_j} \;\; = \;\; \frac{1}{2}|x_j^{e_j}| < |x_j^{e_j}|$$

and hence $x_j^{e_j} + p_j(\mathbf{x}) \neq 0$ in $\mathcal{V}_j$. But $x_j^{e_j} + p_j = g_1 f_1 + \cdots + g_t f_t$ for some $g_1, \ldots, g_t \in \mathcal{R}$, so for each $\mathbf{x} \in \Delta(\mathbf{0}, \mathbf{r}) \setminus \{\mathbf{0}\}$ some $f_i(\mathbf{x})$ must not vanish. With the infinity norm, each $\mathbf{x} \in \Delta(\mathbf{0}, \mathbf{r})$ satisfies $\|\mathbf{x}\| = |x_j| < r_j$ for some j, so $\Delta(\mathbf{0}, \mathbf{r}) \setminus \{\mathbf{0}\} = \cup_{j=1}^{s} \mathcal{V}_j$ which proves the assertion (i). □

Remark on Lemma 5: The form of (iii) implies that if $f_1, \ldots, f_t \subset \mathbb{C}[x_1, \ldots, x_s]$ are polynomials then those $p_1, \ldots, p_t$ are also polynomials in $\mathbb{C}[x_1, \ldots, x_s]$, so Lemma 5 specializes to a lemma about polynomials. Generally in algebra the practice is to give results over as general a field as possible so the ground field could be the rationals or a field of characteristic p and hence the analytic definition above does not apply. Assertion (ii), which is equivalent to the local ring at $\hat{\mathbf{x}} = \mathbf{0}$ being finite dimensional over the ground field, is often used as the working definition of isolated zero in algebraic geometry so in the literature (i) may taken to be identical to (ii). For polynomials the difficulty in this lemma is using the analytic definition of isolated zero above. If it is known that the solution to the system F is finite, then by Hilbert's Nullstellensatz the ring $\mathbb{C}[x_1, \ldots, x_s]/F\mathbb{C}[x_1, \ldots, x_x]$ is finite dimensional over $\mathbb{C}$ and (ii) follows. But in the case where the system F also has positive dimensional components and $\hat{\mathbf{x}}$ is a multiple zero we know of no easier proof in the polynomial case than the one given here. □

In the rest of this section $\mathfrak{M}_s^\alpha$, for $\alpha = 0, 1, 2, \ldots$, will denote the ideal in $\mathbb{C}\{x_1, \ldots, x_s\}$ given by

$$\mathfrak{M}_s^\alpha \;\; = \;\; \{f \in \mathbb{C}\{x_1, \ldots, x_s\} \Big| \mathit{ord}(f) \geq \alpha\}.$$

Of course, $\mathfrak{M}_s^0 = \mathbb{C}\{x_1, \ldots, x_s\}$.

Lemma 6 *The ideal $\mathfrak{M}_s^\alpha$ is generated by the monomials $\mathbf{x}^\mathbf{j}$ with $|\mathbf{j}| = \alpha$, that is, if $f \in \mathfrak{M}_s^\alpha$ then $f = \sum_{|\mathbf{j}|=\alpha} \mathbf{x}^\mathbf{j} g_\mathbf{j}$ where $\mathbf{x}^\mathbf{j}$ and $g_\mathbf{j}$ are elements of $\mathbb{C}\{x_1, \ldots, x_s\}$.*

Proof. The proof goes by induction on s. If $s = 1$, then $f \in \mathfrak{M}_1^\alpha$ is of the form $f = \sum_{k \geq \alpha} a_k x_1^k = x_1^\alpha \sum_{j \geq 0} a_{j+\alpha} x_1^j$ so $\mathfrak{M}_1^\alpha$ is generated by x_1^α.

For $s > 1$, by the Weierstrass Division Theorem [23, Th. 3.3.5] $h = x_s^\alpha$ is a Weierstrass polynomial for x_s. So if $f \in \mathfrak{M}_s^\alpha$ then $f = x_s^\alpha g + q$ where $g \in \mathbb{C}\{x_1, \ldots, x_s\}$ and q is a polynomial in x_s of degree $\alpha - 1$ or less in $\mathbb{C}\{x_1, \ldots, x_{s-1}\}[x_s]$. This means $q = x_s^{\alpha-1} q_1 + \cdots + x_s q_{\alpha-1} + q_\alpha$ where $q_k \in \mathbb{C}\{x_1, \ldots, x_{s-1}\}$. Since f and $x_s^\alpha g$ are in $\mathfrak{M}_s^\alpha$, it follows that $\mathit{ord}(q) \geq \alpha$ so $q_k \in \mathfrak{M}_{s-1}^k$. By induction, each term of q is a sum of monomials in $\mathbb{C}\{x_1, \ldots, x_{s-1}\}$ of degree α multiplied by an element of $\mathbb{C}\{x_1, \ldots, x_{s-1}\}$. Hence f is of the desired form. □

Remark on Lemma 6. This lemma is also needed in the polynomial case if one is to use $\mathbb{C}[[x_1, \ldots, x_s]]$ rather than $\mathbb{C}[x_1, \ldots, x_s]_{\langle x_1, \ldots, x_s \rangle}$ as the local ring (see [4]). In [26], an algebraic proof of the Weierstrass preparation theorem is given for $\mathbb{C}[[x_1, \ldots, x_s]]$ from which the analog of this lemma can be deduced. □

As a consequence of Lemma 5 and Lemma 6, if $\hat{\mathbf{x}} = \mathbf{0}$ is an isolated zero of an analytic system $F = \{f_1, \ldots, f_t\}$ then $\mathbb{C}\{x_1, \ldots, x_s\}/F\mathbb{C}\{x_1, \ldots, x_s\}$ is finite dimensional as a $\mathbb{C}$-algebra (c.f. Theorem 6 part (ii) below). We will also need to refer to a lemma from [19] which uses this fact to define multiplicity as this dimension. For convenience, we paraphrase a special case of that lemma here using our notation.

Lemma 7 (Sommese-Verschelde Local Extension Lemma) [19, Lemma 6] *Let*

$$F \;=\; [f_1, \ldots, f_s]^\top \;:\; \mathbb{C}^s \times \mathbb{C}^t \;\longrightarrow\; C^s$$

be a system of s holomorphic functions. Let $\hat{\mathbf{x}} \in \mathbb{C}^s$ be an isolated solution of $F(\mathbf{x}, \hat{\mathbf{y}}) = \mathbf{0}$ for a fixed $\hat{\mathbf{y}} \in \mathbb{C}^t$. Let $m = \mathit{dim}_{\mathbb{C}}\Big(\mathbb{C}\{x_1, \ldots, x_s\}/F\mathbb{C}\{x_1, \ldots, x_s\}\Big)$. Then there are open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, respectively, such that for any fixed $\mathbf{y} \in \mathcal{V}$ there exist exactly m isolated solutions (counting multiplicities) of the equation $F(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ in $\mathbf{x} \in \mathcal{U}$.

4.2 Equivalence of Hilbert Functions

In §2.1, we defined a Hilbert function

$$\left\{ \begin{array}{rcl} \mathrm{h}(0) & = & \mathit{dim}\Big(\mathcal{D}^0_{\hat{\mathbf{x}}}(F)\Big) \equiv 1 \\ \mathrm{h}(\alpha) & = & \mathit{dim}\Big(\mathcal{D}^\alpha_{\hat{\mathbf{x}}}(F)\Big) - \mathit{dim}\Big(\mathcal{D}^{\alpha-1}_{\hat{\mathbf{x}}}(F)\Big) \quad \text{for } \alpha \in \{1, 2, \ldots\}, \end{array} \right. \tag{58}$$

and showed in Corollary 1 of §2.2 that

$$\mathrm{h}(\alpha) \;=\; \mathit{nullity}\,(S_\alpha) - \mathit{nullity}\,(S_{\alpha-1}) \tag{59}$$

for $\alpha = 1, 2, \cdots$. We now define a new Hilbert function $\mathrm{H}(\alpha)$ and show that it agrees with $\mathrm{h}(\alpha)$.

We will now hold the number of variables s fixed, so we drop the subscript s and just write $\mathfrak{M}^\alpha$ instead of $\mathfrak{M}_s^\alpha$. By Lemma 6 above, each of the ideals $\mathfrak{M}^\alpha$ is finitely generated. In fact, each $\mathfrak{M}^\alpha$ is generated by the $\binom{s+\alpha-1}{s-1}$ monomials $\mathbf{x}^\mathbf{j}$ of total degree α. Note that the projection $\rho_\alpha : \mathbb{C}\{x_1, \ldots, x_s\} \to \mathbb{C}\{x_1, \ldots, x_s\}/\mathfrak{M}^\alpha$ satisfies $\rho_{\alpha+1}(f) = \rho_{\alpha+1}(\mathit{jet}(f, \alpha))$ from which it easily follows that $\mathfrak{M}^\alpha/\mathfrak{M}^{\alpha+1}$ has a basis consisting of the monomials $\mathbf{x}^\mathbf{j}$ of total degree α and so

$$\mathit{dim}\Big(\mathfrak{M}^\alpha/\mathfrak{M}^{\alpha+1}\Big) \;=\; \binom{\alpha + s - 1}{s - 1} \tag{60}$$

as a $\mathbb{C}$-vector space. Now, for a given analytic system $F = \{f_1, \ldots, f_t\}$, we make a filtration $\mathfrak{m}^\alpha$

$$\mathcal{A} \;=\; \mathfrak{m}^0 \;\supseteq\; \mathfrak{m}^1 \;\supseteq\; \mathfrak{m}^2 \;\supseteq\; \cdots$$

on $\mathcal{A} = \mathbb{C}\{x_1, \ldots, x_s\}/F\mathbb{C}\{x_1, \ldots, x_s\}$ by defining $\mathfrak{m}^\alpha$ to be the image of $\mathfrak{M}^\alpha$ in $\mathcal{A}$. It is not hard to see that $\mathfrak{m}^\alpha$ is naturally isomorphic to $\mathfrak{M}^\alpha/(\mathfrak{M}^\alpha \cap F\mathbb{C}\{x_1, \ldots, x_s\})$ as a $\mathbb{C}$-vector space. Furthermore, $\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$ is a quotient of $\mathfrak{M}^\alpha/\mathfrak{M}^{\alpha+1}$ and is thus a $\mathbb{C}$-vector space of dimension less than or equal to $\binom{\alpha+s-1}{s-1}$.

We define the Hilbert function $\mathrm{H}(\alpha)$ by

$$\begin{cases} \mathrm{H}(0) &=& \mathit{dim}\Big(\mathcal{A}/\mathfrak{m}^1\Big) \equiv 1 \\ \mathrm{H}(\alpha) &=& \mathit{dim}\Big(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}\Big) \quad \text{for } \alpha \in \{1, 2, \ldots\}, \end{cases} \tag{61}$$

and proceed to establish the following lemma.

Lemma 8 *Let $F = \{f_1, \ldots, f_t\}$ be a system of analytic functions on an open subset of $\mathbb{C}^s$ containing $\hat{\mathbf{x}} = \mathbf{0}$ such that $\hat{\mathbf{x}}$ is a zero of the system. Then*

$$H(\alpha) \;=\; h(\alpha)$$

for all $\alpha = 0, 1, 2, \ldots$.

Proof. Let $\mathcal{I} = F\mathbb{C}\{x_1, \ldots, x_s\}$ then $\mathcal{A} = \mathbb{C}\{x_1, \ldots, x_s\}/\mathcal{I}$ as above. For $g \in \mathbb{C}\{x_1, \ldots, x_s\}$, write $g = g^{(0)} + g^{(1)} + g^{(2)} + \cdots$ where $g^{(\lambda)}$ is the sum of all terms of total degree λ. We define the function In_α by

$$\mathit{In}_\alpha(g) \;=\; \begin{cases} g^{(\alpha)} & \text{if } g^{(\lambda)} = 0 \text{ for } \lambda < \alpha \text{ and } g^{(\alpha)} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For the ideal $\mathcal{J}$ we write $\mathit{In}_\alpha(\mathcal{J}) = \{\mathit{In}_\alpha(f) \,\big|\, f \in \mathcal{J}\}$. It then follows a commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & \mathfrak{M}^{\alpha+1} \cap \mathcal{I} & \longrightarrow & \mathfrak{M}^\alpha \cap \mathcal{I} & \xrightarrow{\mathit{In}_\alpha} & \mathit{In}_\alpha(\mathcal{I}) & \longrightarrow & \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & \mathfrak{M}^{\alpha+1} & \longrightarrow & \mathfrak{M}^\alpha & \xrightarrow{\mathit{In}_\alpha} & \mathit{In}_\alpha(\mathfrak{M}^\alpha) & \longrightarrow & \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & \mathfrak{m}^{\alpha+1} & \longrightarrow & \mathfrak{m}^\alpha & \longrightarrow & \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1} & \longrightarrow & \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & &
\end{array}$$

where the unmarked linear maps are the obvious inclusions or projections except for the last column. In the last column, $\mathit{In}_\alpha(\mathcal{I})$ consists of forms of degree α which are initial forms of elements of $\mathcal{I}$ and $\mathit{In}_\alpha(\mathfrak{M}^\alpha)$ is just the space of all forms of degree α. So $\mathit{In}_\alpha(\mathcal{I}) \to \mathit{In}_\alpha(\mathfrak{M}^\alpha)$ is again an

inclusion. The map $In_\alpha(\mathfrak{M}^\alpha) \to \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$ is induced by the projection $\mathfrak{M}^\alpha \to \mathfrak{m}^\alpha$, i.e. the unique map making the bottom right square of the diagram commute.

The rows of this diagram are all exact (image in is kernel out) as are the first two columns. Exactness of the third column then follows from the 3×3-lemma of homological algebra. Because of this third column exactness, we have

$$H(\alpha) \;=\; dim\big(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}\big) \;=\; dim_{\mathbb{C}}(In_\alpha(\mathfrak{M}^\alpha)) - dim_{\mathbb{C}}(In_\alpha(\mathcal{I})) \;=\; \binom{\alpha-1+s}{s-1} - dim_{\mathbb{C}}(In_\alpha(\mathcal{I})). \quad (62)$$

Thus, to identify $H(\alpha)$, we only need to calculate $dim_{\mathbb{C}}\big(In_\alpha(\mathcal{I})\big)$, and we shall do so by using the Macaulay matrix S_α. Write $jet(\mathcal{I},\alpha)$ for the vector space spanned by $\{jet(g,\alpha) | g \in \mathcal{I}\}$. Since $\hat{\mathbf{x}} = \mathbf{0}$, the entry in the row labeled $\mathbf{x}^{\mathbf{k}} f_i$ and the column indexed $\mathbf{x}^{\mathbf{j}}$ is the coefficient of $\mathbf{x}^{\mathbf{j}}$ of the polynomial $\mathbf{x}^{\mathbf{k}} f_i$, thus $jet(\mathcal{I},\alpha)$ is the rowspace of S_α, which is row equivalent to a matrix in a reduced row echelon form with linearly independent rows:

$$S_\alpha \;\simeq\; A_\alpha \;=\; \left[\begin{array}{c|c} rowspace\ S_{\alpha-1} & B_\alpha \\ \hline \mathbf{0} & C_\alpha \end{array}\right]. \quad (63)$$

Each row of C_α corresponds to an element of $In_\alpha(\mathcal{I})$ by multiplying each entry by its column index and adding. These elements clearly form a basis for $In_\alpha(\mathcal{I})$.

The total number of rows in A_α is $rank(S_\alpha)$ while the number of rows of B_α is $rank(S_{\alpha-1})$. So the number of rows in C_α is

$$\begin{aligned} dim_{\mathbb{C}}\big(In_\alpha(\mathcal{I})\big) &= rank(S_\alpha) - rank(S_{\alpha-1}) \\ &= \left(\binom{\alpha+s}{s} - nullity(S_\alpha)\right) - \left(\binom{\alpha-1+s}{s} - nullity(S_{\alpha-1})\right) \\ &= \binom{\alpha-1+s}{s-1} - \mathrm{h}(\alpha) \end{aligned} \quad (64)$$

where the difference of binomial coefficients comes from either Pascal's identity or noting that the number of monomials of total degree less than or equal to α minus those of total degree less than or equal to $\alpha-1$ is the number of monomials of total degree exactly α. Identity (59) is also applied for the third equality. The lemma follows by combining equations (62) and (64). □

Remark: From an algebraic-geometric point of view, the rings $\mathbb{C}\{x_1,\ldots,x_s\}$ and $\mathcal{A} = \mathbb{C}\{x_1,\ldots,x_s\}/F\mathbb{C}\{x_1,\ldots,x_s\}$ are *local rings* with $\mathfrak{M}^1$ and $\mathfrak{m}^1$ as their respective unique maximal ideals. The ideals $\mathfrak{M}^\alpha$ and $\mathfrak{m}^\alpha$ are powers of these maximal ideals. The *associated graded ring* $\mathrm{Gr}(\mathcal{A}) = \bigoplus \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$ is also known as the *tangent cone* (see [8, 10, 26]). The associated graded ring $\mathrm{Gr}(\mathcal{A})$ is, in particular, a *standard graded algebra* and a Hilbert function $\mathrm{H}(\alpha)$ is defined as in (61) [10, p. 185]. Much is known about the behavior of such Hilbert functions [8, 10, 20]. For example, if for some fixed β one has $H(\beta) = \gamma \le \beta$, then $H(\alpha) \le \gamma$ for all $\alpha \ge \beta$. We shall start §4.3 with a proof of this in the case $\gamma = 0$. □

Example 7 Consider the system $F = \{\sin(x-y) + x^3,\ x - y + \sin^3 y\}$. We calculate S_4 and reduce this exact matrix to reduced row echelon form with column headings provided:

1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	x^4	x^3y	x^2y^2	xy^3	y^4
0	1	−1	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	1	0	−1	0	0	0	0	0	0	0	0	2
0	0	0	0	1	−1	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	1	0	0	−1	0	0	0	0	0
0	0	0	0	0	0	0	1	0	−1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1
0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1
0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1

where the rectangular boxes show the matrices C_α corresponding to the initial forms of degrees 1,2,3 and 4. Thus, for example, the initial forms of degree 3 are spanned by $x^3 - y^3$, $x^2y - y^3$ and $xy^2 - y^3$. In each case the initial form matrix is of full row rank with one more column than row, implying $\mathrm{h}(1) = \mathrm{h}(2) = \mathrm{h}(3) = \mathrm{h}(4) = 1$ while $\mathrm{h}(0)$ is always equal to 1. Consequently, the reduced row echelon form of S_4 is sufficient to reveal that the multiplicity is at least 5. Now, the reduced row echelon form of S_5 shows the above matrix in the upper 10×15 block and the identity matrix in the lower right 6×6 block with zeros elsewhere, which implies $\mathrm{h}(5) = 0$. In §4.3 below we shall see that the reduced row echelon form of S_5 reveals the multiplicity structure completely for this system. □

4.3 The Local Finiteness, Depth and Multiplicity Consistency Theorems.

We need one additional lemma which is a special case of Nakayama's Lemma [4, 8].

Lemma 9 *Assume $F = \{f_1, \ldots, f_t\}$ is a system of functions that are analytic in a neighborhood of their common zero $\hat{\mathbf{x}} = \mathbf{0}$ in $\mathbb{C}^s$. Let $\mathcal{I} = F\mathbb{C}\{x_1, \ldots, x_s\}$, $\mathcal{A} = \mathbb{C}\{x_1, \ldots, x_s\}/\mathcal{I}$ and let $\mathfrak{M}^\alpha$ be the ideal of series of order α or greater. Then there is a filtration*

$$\mathcal{A} \quad = \quad \mathfrak{m}^0 \supseteq \mathfrak{m}^1 \supseteq \mathfrak{m}^2 \supseteq \ldots \tag{65}$$

given by $\mathfrak{m}^\alpha = \mathfrak{M}^\alpha/(\mathfrak{M}^\alpha \cap \mathcal{I})$. Let $\mathrm{h}(\alpha)$ be one of the equivalent Hilbert functions defined in (58) *and* (61)*. The following assertions hold:*

(i) *If for some $\beta > 0$, $\mathfrak{m}^\beta = \mathfrak{m}^{\beta+1}$ then $\mathfrak{M}^\beta \subseteq \mathcal{I}$.*
(ii) *If $\mathrm{h}(\beta) = 0$, then $\mathrm{h}(\alpha) = 0$ for all $\alpha \geq \beta$.*

Proof. By Lemma 6, the ideal $\mathfrak{m}^\beta$ as a quotient of $\mathfrak{M}^\beta$ is generated by monomials of total degree β in $x_1, \ldots, x_s$. For convenience, we list these monomials in some order $a_1, \ldots, a_n$ where $n = \binom{\beta+s-1}{s-1}$. Likewise, $\mathfrak{m}^{\beta+1}$ is generated by monomials of total degree $\beta + 1$, that is, monomials of the non-unique form $x_i a_j$. Assuming $\mathfrak{m}^\beta = \mathfrak{m}^{\beta+1}$, each a_j is a sum of elements in $\mathbb{C}\{x_1, \ldots, x_s\}$ multiplied by $x_i a_j$. Collecting a_j, we have

$$a_k \quad = \quad c_{k,1}a_1 + c_{k,2}a_2 + \cdots + c_{k,n}a_n$$

in the ring $\mathcal{A}$. Note that each $c_{k,j}$ is represented by an element of $\mathfrak{m}^1$. We then obtain a system of equations in $\mathcal{A}$

$$\begin{aligned} 0 &= (c_{1,1}-1)a_1 + c_{1,2}a_2 + \cdots + c_{1,n}a_n \\ 0 &= c_{2,1}a_1 + (c_{2,2}-1)a_2 + \cdots + c_{2,n}a_n \\ \vdots &\quad \vdots \\ 0 &= c_{n,1}a_n + \cdots + c_{n,n-1}a_{n-1} + (c_{n,n}-1)a_n. \end{aligned}$$

However, each $(c_{\alpha,\alpha}-1)$ is invertible in $\mathbb{C}\{x_1,\ldots,x_s\}$ because it is a convergent series with constant term -1 and so has a reciprocal convergent on a small open set about $\mathbf{0}$, yielding that $(c_{\alpha,\alpha}-1)$ is also invertible in the quotient ring $\mathcal{A}$. More generally, any sum of an invertible element with an element of $\mathfrak{m}^1$ is invertible. Thus the determinant of the system is clearly invertible. It follows that the system is nonsingular so each $a_k = 0$ in $\mathcal{A}$. This means a_k represents some element in $\mathcal{I}$. Thus $\mathfrak{M}^\beta \subseteq \mathcal{I}$, proving assertion (i).

For assertion (ii), note by Lemma 8, $\mathrm{h}(\beta) = \dim_{\mathbb{C}}\big(\mathfrak{m}^\beta/\mathfrak{m}^{\beta+1}\big)$. So if $\mathrm{h}(\beta)=0$ then $\mathfrak{m}^\beta = \mathfrak{m}^{\beta+1}$ and by assertion (i), the ideal $\mathfrak{M}^\beta \subseteq \mathcal{I}$. Since we have a descending filtration, hence $\mathfrak{M}^\alpha \subseteq \mathfrak{M}^\beta \subseteq \mathcal{I}$ for any $\alpha \geq \beta$. Consequently, $\mathfrak{m}^\alpha = \mathfrak{M}^\alpha/(\mathfrak{M}^\alpha \cap \mathcal{I}) = \mathfrak{M}^\alpha/\mathfrak{M}^\alpha = 0$ in $\mathcal{A}$. Thus $\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1} = 0$ and $\mathrm{h}(\alpha) = 0$ by Lemma 8. □

The following is a more detailed version of Theorem 1.

Theorem 6 (Local Finiteness Theorem) *Let $\mathcal{R}$ be the ring of analytic functions on an open set $\mathcal{U} \subseteq \mathbb{C}^s$ and assume $\hat{\mathbf{x}} = \mathbf{0} \in \mathcal{U}$ is a zero of the system $F = \{f_1,\ldots,f_t\} \subset \mathcal{R}$. Let $\mathrm{h}(\alpha)$ be the Hilbert function defined by equation (13). Then the following are equivalent:*

(i) *The zero $\hat{\mathbf{x}}$ of the system F is isolated.*

(ii) *The local ring $\mathcal{A} = \mathbb{C}\{x_1,\ldots,x_s\}/F\mathbb{C}\{x_1,\ldots,x_s\}$ is finite dimensional $\mathbb{C}$-vector space.*

(iii) $\sum_{\alpha\geq 0}\mathrm{h}(\alpha) < \infty$.

(iv) *The dual space $\mathcal{D}_{\hat{\mathbf{x}}}(F)$ is finite-dimensional, i.e. Theorem 1 holds.*

(v) *For a sufficiently large α, the Macaulay matrix S_α is row equivalent to a matrix $\begin{bmatrix} R & B \\ 0 & C \end{bmatrix}$ where C is the $n \times n$ identity matrix with $n = \binom{\alpha+s-1}{s-1}$.*

(vi) *For any monomial $\mathbf{x}^{\mathbf{j}}$ of sufficiently large total degree $|\mathbf{j}|$, there is a $p_{\mathbf{j}} \in \mathcal{R}$ with $\mathit{ord}(p_{\mathbf{j}}) > |\mathbf{j}|$ such that $\mathbf{x}^{\mathbf{j}} + p_{\mathbf{j}} \in F\mathbb{C}[x_1,\ldots,x_s]$.*

Proof. (i) ⇒ (ii): By Lemma 5, for each $j \in \{1,\ldots,s\}$, there is an integer e_j with $x^{e_j} \in F\mathbb{C}\{x_1,\ldots,x_s\}$. If $\beta = \max\{e_j\}$, then any monomial $\mathbf{x}^{\mathbf{j}}$ of total degree β is contained in $F\mathbb{C}\{x_1,\ldots,x_s\}$. That is, with the notation of Lemma 6, we have $\mathfrak{M}^\alpha \subseteq F\mathbb{C}\{x_1,\ldots,x_s\}$. Thus the filtration (65) terminates at or before β because $\mathfrak{m}^\beta = 0$. However

$$\begin{aligned} \dim_{\mathbb{C}}(\mathcal{A}) &= \dim_{\mathbb{C}}(\mathcal{A}) - \dim_{\mathbb{C}}\big(\mathfrak{m}^\beta\big) \\ &= \Big(\dim_{\mathbb{C}}(\mathcal{A}) - \dim_{\mathbb{C}}\big(\mathfrak{m}^1\big)\Big) + \Big(\dim_{\mathbb{C}}\big(\mathfrak{m}^1\big) - \dim_{\mathbb{C}}\big(\mathfrak{m}^2\big)\Big) + \cdots + \Big(\dim_{\mathbb{C}}\big(\mathfrak{m}^{\beta-1}\big) - \dim_{\mathbb{C}}\big(\mathfrak{m}^\beta\big)\Big) \\ &= \dim_{\mathbb{C}}\big(\mathfrak{m}^0/\mathfrak{m}^1\big) + \dim_{\mathbb{C}}\big(\mathfrak{m}^1/\mathfrak{m}^2\big) + \cdots + \dim_{\mathbb{C}}\big(\mathfrak{m}^{\beta-1}/\mathfrak{m}^\beta\big) < \infty \quad (66) \end{aligned}$$

since each $\dim_{\mathbb{C}}\big(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}\big) < \binom{\alpha+s-1}{s-1}$.

(ii) $\Rightarrow$ (iii): Equation (66) shows that

$$\mathit{dim}_{\mathbb{C}}\big(\mathcal{A}\big) \;=\; \sum_{\alpha\geq 0} \mathit{dim}_{\mathbb{C}}\big(\mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1}\big) \;=\; \sum_{\alpha\geq 0} h(\alpha). \tag{67}$$

Assertion (iii) thus follows the assumption that $\mathit{dim}_{\mathbb{C}}(\mathcal{A}) < \infty$.

(iii) $\Leftrightarrow$ (iv): Since $\mathit{dim}\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big) = \sum_{\alpha\geq 0} \mathrm{h}(\alpha)$ by equation (13), assertions (iii) and (iv) are equivalent.

(iii) $\Rightarrow$ (v): Because $\sum_{\alpha\geq 0} \mathrm{h}(\alpha) < \infty$ is a sum of nonnegative integers, clearly $\mathrm{h}(\alpha) = 0$ for sufficiently large α's. Moreover, the equation (64) leads to $\mathit{dim}_{\mathbb{C}}\big(\mathit{In}_{\alpha}(\mathcal{I})\big) = \binom{\alpha+s-1}{s-1}$ and the matrix C_α in (63) is thus a square matrix with linearly independent rows, implying C_α can be transformed to the identity matrix by row operations.

(v) $\Rightarrow$ (vi): By construction, since $\hat{\mathbf{x}} = \mathbf{0}$, each row of S_α consists of the coefficients of $\mathit{jet}(\mathbf{x}^{\mathbf{j}} f_i, \alpha)$ for some monomial in $x_1, \ldots, x_s$. In the row equivalent matrix, rows are then linear combinations of such jets. Since the jet operator is linear for fixed α, the row with a 1 in the column indexed $\mathbf{j}$ consists of the coefficients of the jet of $\mathbf{x}^{\mathbf{j}} + p_{\mathbf{j}} \in F\mathbb{C}[x_1, \ldots, x_s]$ with $ord(p_{\mathbf{j}}) > \alpha$.

(vi) $\Rightarrow$ (i): By assertion (vi), there exists an α such that for each monomial of the form x_j^α there is $x_j^\alpha + p_j \in F\mathbb{C}[x_1, \ldots, x_s]$. Therefore assertion (iii) of Lemma 5 is satisfied and hence $\hat{\mathbf{x}}$ is an isolated zero. □

Theorem 7 (Depth Theorem) *Let $F = \{f_1, \ldots, f_t\}$ be a system of analytic functions in an open set of $\mathbb{C}^s$ with an isolated zero $\hat{\mathbf{x}} = \mathbf{0}$. Then there is a positive integer $\delta = \delta_{\hat{\mathbf{x}}}(F)$ called the* depth of the isolated zero $\hat{\mathbf{x}}$ *and the following are equivalent:*

(i) *δ is the largest integer with $\mathrm{h}(\delta) \neq 0$.*

(ii) *δ is the smallest integer with $\mathrm{h}(\delta + 1) = 0$.*

(iii) *δ is the highest differential order of a functional in $\mathcal{D}_{\hat{\mathbf{x}}}(F)$.*

(iv) *δ is the smallest integer so that the Macaulay matrix $S_{\delta+1}$ is row equivalent to a matrix $\begin{bmatrix} R & B \\ 0 & C \end{bmatrix}$ where C is the $n \times n$ identity matrix with $n = \binom{\delta+s}{s-1}$.*

Proof. (i) $\Leftrightarrow$ (ii): By Lemma 9, once $\mathrm{h}(\alpha) = 0$ it remains zero.

(i) $\Leftrightarrow$ (iii): This is equation (13)

(i) $\Leftrightarrow$ (iv): This follows from equations (63) and (64). □

We can now prove the more detailed Multiplicity Consistency Theorem below.

Theorem 8 (Multiplicity Consistency Theorem) *Suppose $\hat{\mathbf{x}} = \mathbf{0}$ is an isolated zero of the analytic system $F = \{f_1, \ldots, f_t\}$ defined on an open set in $\mathbb{C}^s$. If F is a square system, i.e. $t = s$, and analytic, then the following four formulations of the* multiplicity m of the system F at the zero $\hat{\mathbf{x}}$ *are all equivalent:*

(i) $m \;=\; \mathit{dim}_{\mathbb{C}}\Big(\mathbb{C}\{x_1, \ldots, x_s\}/F\mathbb{C}\{x_1, \ldots, x_s\}\Big)$.

(ii) $m = \sum_{\alpha \geq 0} \mathrm{h}(\lambda)$ *where* $\mathrm{h}(\cdot)$ *is the Hilbert function defined in* (13).

(iii) $m = \mathit{dim}\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big)$.

(iv) $m = \mathit{dim}\big(\mathbb{C}[x_1, \ldots, x_s]/F\mathbb{C}[x_1, \ldots, x_s]\big)$ *if* $F \in \mathbb{C}[x_1, \ldots, x_s]$ *is a system of polynomials.*

Proof. The equivalence of (i) and (ii) comes from (67), while the equivalence of (i) and (iii) follows by combining equations (67) and (13).

If the polynomial system F is zero-dimensional (i) is equivalent to (iv) by [4, Prop. 2.11]. Otherwise in [8, Prop 5.5.12] a calculation of the associated graded ring of $\mathbb{C}[x_1, \ldots, x_s]/F\mathbb{C}[x_1, \ldots, x_s]$ in terms of the initial ideal is given which is essentially identical to our argument in the proof of Lemma 8. Equation (67) shows that the $\mathbb{C}$-dimension of a zero dimensional local ring is the same as the $\mathbb{C}$-dimension of its associated graded ring which proves (i) $\Leftrightarrow$ (iv). □

The Multiplicity Consistency Theorem immediately leads to the following corollaries.

Corollary 2 *The Perturbation Invariance Theorem* (Theorem 3) *holds.*

Proof. A straightforward verification using the equivalence of (i) and (ii) in Theorem 8 and the Local Extension Lemma (Lemma 7). □

Corollary 3 *Under the assumptions of Theorem 8, the multiplicity*

$$m = \mathit{dim}\big(\mathcal{D}_{\hat{\mathbf{x}}}(\mathit{jet}(F, \delta + 1))\big)$$

where δ *is the depth* $\delta_{\hat{\mathbf{x}}}(F)$ *of the system* F *at* $\hat{\mathbf{x}}$ *and* $\mathit{jet}(F, \delta + 1)$ *is the polynomial system* $\{\mathit{jet}(f_1, \delta + 1), \ldots, \mathit{jet}(f_t, \delta + 1)\}$.

Proof. Since the Macaulay matrix S_α of F is identical to that of the polynomial system $\mathit{jet}(F, \alpha)$ for $\alpha \leq \delta + 1$. By (iv) of Theorem 7 $\mathit{jet}(F, \delta + 1)$ has depth δ and so by (iii) above the multiplicity of the polynomial system $\mathit{jet}(F, \delta + 1)$ is $\mathit{dim}\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big)$ and equals the multiplicity of F defined in Definition 1. □

Remark: In commutative algebra the term *regularity index* or just *index* is used instead of our *depth*. Specifically, the regularity index is the first β such that $\mathrm{H}(\alpha) = \mathrm{HP}(\beta)$ for all $\alpha \geq \beta$ where $\mathrm{HP}(\cdot)$ is the Hilbert Polynomial [10]. In the case of an isolated zero the Hilbert Polynomial is identically zero so the regularity index of $\hat{\mathbf{x}}$ is $\delta_{\hat{\mathbf{x}}}(F) + 1$. □

Acknowledgements. The authors wish to thank following scholars: Along with many insightful discussions, Andrew Sommese provided a preprint [2] which presented an important application of this work, Hans Stetter provided the diploma thesis [24] of his former student, Teo Mora pointed out Macaulay's original contribution [14] elaborated in his book [16], and Lihong Zhi pointed out the reference [12].

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