The Numerical Factorization of Polynomials

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Abstract Polynomial factorization in conventional sense is an ill-posed problem due to its discontinuity with respect to coefficient perturbations, making it a challenge for numerical computation using empirical data. As a regularization, this paper formulates the notion of numerical factorization based on the geometry of polynomial spaces and the stratification of factorization manifolds. Furthermore, this paper establishes the existence, uniqueness, Lipschitz continuity, condition number, and convergence of the numerical factorization to the underlying exact factorization, leading to a robust and efficient algorithm with a Matlab implementation capable of accurate polynomial factorizations using floating point arithmetic even if the coefficients are perturbed.

1 Introduction

Polynomial factorization is one of the fundamental algebraic operations in theory and in applications. It is also an enduring research subject in the field of computer algebra as well as a significant success of symbolic computation (c.f. the survey [17]). Factorization functionalities have been standard features of computer algebra systems such as Maple and Mathematica with a common assumption that the

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coefficients are represented exactly. Nonetheless, theoretical advancement and algorithmic development are still in early stages in many cases. When a polynomial is approximately known with a limited accuracy in coefficients, the very meaning of its factorization as we know it becomes a question, as illustrated in the following example. More precisely, a well-posed notion of numerical factorization has not been established, leaving a gap in the foundation of its computation.

Example 1 We illustrate the central question of this paper: *Assume the polynomial*

$$f \ = \ x^2 y + 0.857143\, xy^2 + 0.833333\, x^2 + 1.38095\, xy + 0.571429\, y^2 + 0.555556\, x + 0.476190\, y \quad (1)$$

is given as the empirical data of a factorable polynomial $\tilde{f}$. *Knowing that the data are imperfect with an error bound* $\|f - \tilde{f}\| \le 10^{-5}$, *what is the factorization of the underlying polynomial* $\tilde{f}$?

The factorization of f in conventional sense doesn't exist while the underlying polynomial $\tilde{f}$ is factorable but not known exactly. Intuitively, one can ask a more modest question: *Is there a factorable polynomial near* f *within the data error bound* 10^{-5}? This latter question is similar to an open problem in [16] and the answer is ambiguous: The polynomial f is near many factorable polynomials, as shown in Table 1.

factorable polynomial near f	distance
$\tilde{f} \ = \ (x + \frac{2}{3})(y + \frac{5}{6})(x + \frac{6}{7}\, y)$ $\longleftarrow$ underlying polynomial	2.53×10^{-6}
$\hat{f} \ = \ 0.9999994\, (x + 0.6666667)(y + 0.8333333)(x + 0.8571429\, y)$	2.11×10^{-6}
$\hookleftarrow$ the numerical factorization	
$f_1 \ = \ 1.0000002\, (y + 0.8333327)(x^2 + 0.6666663\, x + 0.5714287\, y + 0.8571420\, xy)$	1.75×10^{-6}
$f_2 \ = \ 1.0000002\, (x + 0.8571425\, y)(xy + 0.8333321\, x + 0.6666661\, y + 0.5555555)$	1.60×10^{-6}
$f_3 \ = \ 0.9999997\, (x + 0.66666728)(xy + 0.83333331\, x + 0.7142836\, y + 0.8571432\, y^2)$	8.70×10^{-7}
$\hookleftarrow$ the nearest factorable polynomial	

Table 1 The polynomial f in (1) is near many factorable polynomials with various distances.

The *numerical factorization* of f within the error tolerance 10^{-5}, as we shall define in §5, is the exact factorization of $\hat{f}$ in Table 1 and accurately approximates the factorization of $\tilde{f}$ from which f is constructed by rounding up digits. The nearest polynomial to the data f, however, is not $\hat{f}$ but f_3 whose factorization does *not* resemble that of $\tilde{f}$. In fact, the factorable polynomial with the smallest distance to the data is almost certain to have an incorrect factorization structure by the Factorization Manifold Embedding Theorem in §4 whenever the underlying polynomial $\tilde{f}$ has more than two factors. $\square$

As shown in this example, conventional factorization is a so-called *ill-posed problem* for numerical computation since the factorization is discontinuous with respect to data perturbations. Consequently, fundamental questions arise such as if, under what conditions, by computing which factorization and to what accuracy we can recover the factorization from empirical data. In this paper, we establish the geometry of the polynomial (topological) spaces in Factorization Manifold Theorem and Factorization Manifold Embedding Theorem. Based on the geometry we rigorously formulate the notion of the *numerical factorization*. We prove the so-defined numerical factorization eliminates the ill-posedness of the conventional factorization and accurately approximates the intended exact factorization (Numerical Factorization Theorem) with a finite sensitivity measure that is conveniently attainable (Numerical Factorization Sensitivity Theorem). As a result, the ill-posed factorization problem in numerical computation is completely regularized

as a well-posed numerical factorization problem that approximates the intended factorization with an accuracy in the same order of the data precision.

Our results can be narrated as follows. The collection of polynomials possessing a nontrivial factorization structure is a complex analytic manifold of a positive codimension and every such manifold is embedded in the closures of certain manifolds of lower codimensions. This dimension deficit provides a singularity measurement of polynomials on the manifold and fully explains the ill-posedness their factorizations: An infinitesimal perturbation reduces the singularity and pushes a polynomial away from its native manifold into the open dense subset of polynomials with a trivial factorization structure, making the exact factorization on the empirical data meaningless. Based on the geometric analysis, we formulate the notion of the numerical factorization as the exact factorization of the polynomial on the nearby factorization manifold of the highest singularity having the smallest distance to the data. Under the assumption that the data error is small, the original factorization can be recovered accurately by the numerical factorization of the data polynomial within a proper error bound even if it is perturbed. From the Tubular Neighborhood Theorem in differential geometry, the numerical factorization is a well posed problem as it uniquely exists, is Lipschitz continuous and approximates the exact factorization of the underlying polynomial the data represent. The accuracy of the recovered factorization is in the same order of the data accuracy since the factorization is Lipschitz continuous on that manifold. Moreover, the conventional factorization becomes a special case of the numerical factorization within a small error tolerance. Although a specific algorithm is not proposed here, the geometric analysis naturally leads to a two-staged computing strategy for the numerical factorization: Identifying the factorization manifold by a squarefree factorization and a proper reducibility test, followed by the Gauss-Newton iteration [2, 6, 26, 30] for minimizing the distance to the factorization manifold.

This paper attempts to bridge differential geometry, computer algebra and numerical analysis. As an effective analytical tool that still appears to be underused, geometry has led to many penetrating insights in numerical analysis (e.g. [3, 14]) and effective algorithms such as homotopy methods based on Sard's Theorem and Theorem of Bertini (e.g. [1, 22]). Polynomial factorization problem has been studied from geometric perspective such as in [4, 5, 8]. This paper broadens the geometric analysis into a numerical computation of a basic problem in computer algebra by establishing the stratified complex analytic manifolds of factorization and their tubular neighborhood. In a seminal technical report [14], Kahan is the first to discover the hidden continuity on manifolds for generally discontinuous solutions of ill-posed algebraic problems. Recent works such as [30, 31] made progress along this directions. This work provides a complete regularization of a typical ill-posed algebraic problem in numerical polynomial factorization by establishing its existence, uniqueness, Lipschitz continuity, convergence and condition number. Regularizations to this extent should now be expected for other ill-posed algebraic problems that share a similar geometry.

For exact polynomial factorization, many effective methods have been developed over the past several decades. Those algorithms and complexity analyses have been studied extensively. The work of Sasaki Suzuki, Kolar and Sasaki [24] introduces the techniques of extended Hensel construction and the trace recombination that lead to factorization algorithms such as van Hoeij's trace recombina-

tion [12] for univariate polynomial factorization of integer coefficients. The first polynomial-time factorization algorithms is given by Lenstra, Lenstra and Lovasz [21] for univariate polynomial factorization, and by Kaltofen and Von zur Gathen [11,15] for multivariate polynomials. Rigorous proofs are also provided in these works on the probabilities and the complexities. At present, the algorithm having the lowest complexity for exact bivariate polynomial factorization appears to be due to Lecerf [20].

Many authors made pioneer contributions to the numerical factorization problem of multivariate polynomials, such as pseudofactors by Huang, Stetter, Wu and Zhi [13], the numerical reducibility tests by Galligo and Watt [7] and by Kaltofen and May [18], computing zero sum relations by Sasaki [25], interpolating the irreducible factors as curves by Corless, Giesbrecht, Van Heij, Kotsireas and Watt [4], and by Corless, Galligo, Kotsireas and Watt [5]. Finding a nearby factorable polynomial as proposed in [4,7,8,10,16,17,18] has played an indispensable role in the advancement of numerical polynomial factorization, even though such a backward accuracy alone is insufficient in numerical factorizations as illustrated in Example 1. In [27], Sommese, Verschelde and Wampler developed a homotopy continuation method along with monodromy grouping, and Verschelde released and has maintained the first numerical factorization software as part of the PHC package [29] for solving polynomial systems. A breakthrough due to Ruppert's differential forms [23] led to a novel hybrid factorization algorithm [9] by Gao, and the development of a numerical factorization algorithm by Gao, Kaltofen, May, Yang and Zhi in [10,19]. Based on the formulation and analysis of this paper, we implemented a preliminary testing algorithm as a Matlab module that shares a root similar to [9,10,19] along with several new developments for demonstration of theories in this paper.

The results of this paper is not limited to multivariate polynomials. The numerical factorization theory and computational strategy extend to the univariate polynomial factorization, which is also known as polynomial root-finding where a recent major development enables accurate computation of multiple roots without extending the hardware precision even if the coefficients are perturbed [30]. This paper provides a unified framework for the numerical factorization including the univariate factorization as a special case.

2 Preliminaries

We consider polynomials in variables $x_1, \ldots, x_\ell$ with coefficients in the field $\mathbb{C}$ of complex numbers. The ring of these polynomials is commonly denoted by $\mathcal{R} = \mathbb{C}[x_1, \ldots, x_\ell]$. The ℓ-**tuple degree** of a polynomial f is defined as a vector $deg(f) = \big(deg_{x_1}(f), \ldots, deg_{x_\ell}(f)\big)$ where $deg_{x_j}(f)$ is the degree of f in x_j. For any ℓ-tuple degree $\mathbf{n}$, denote

$$\mathcal{R}_{\mathbf{n}} := \big\{ p \in \mathcal{R} \mid deg(p) \leq \mathbf{n} \big\}$$
$$\mathscr{R}_{\mathbf{n}} := \big\{ p \in \mathcal{R} \mid deg(p) = \mathbf{n} \big\}.$$

Here $\mathcal{R}_{\mathbf{n}}$ is a vector space whose dimension is denoted by $\langle \mathbf{n} \rangle$. Inequalities between ℓ-tuple degrees are componentwise. With a monomial basis in lexicographical order, a polynomial $f = f_1 x^{\mathbf{m}_1} + f_2 x^{\mathbf{m}_2} + \cdots + f_{\langle \mathbf{n} \rangle} x^{\mathbf{m}_{\langle \mathbf{n} \rangle}}$ in $\mathcal{R}_{\mathbf{n}}$ corresponds to a unique **coefficient vector** denoted by $[\![f]\!] := (f_1, \ldots, f_{\langle \mathbf{n} \rangle}) \in \mathbb{C}^{\langle \mathbf{n} \rangle}$,

such as the polynomial $f = 3\,x_1^2 x_2 - 4\,x_1 x_2 + 5\,x_1 + 6 \in \mathcal{R}_{(2,1)}$ corresponding to $[\![f]\!] = (3, 0, -4, 5, 0, 6) \in \mathbb{C}^6$. A subset $\Omega \subset \mathcal{R}_{\mathbf{m}}$ corresponds to the subset $[\![\Omega]\!] = \{[\![p]\!] \in \mathbb{C}^{\langle \mathbf{m} \rangle} \mid p \in \Omega\}$ in $\mathbb{C}^{\langle \mathbf{m} \rangle}$. Here $\mathbb{C}^n$ is the vector space of n-dimensional vectors of complex numbers. All vectors in this paper are ordered arrays denoted by boldface lowercase letters or in the form of $[\![\cdot]\!]$. The Euclidean norm in $\mathbb{C}^{\langle \mathbf{n} \rangle}$ induces the polynomial norm as $\|f\| := \big\|[\![f]\!]\big\|_2$, making $\mathcal{R}_\mathbf{n}$ a topological metric space.

There are no differences between factoring a polynomial and factoring its nonzero constant multiple. We say p and q are **equivalent**, denoted by $p \sim q$, if $p = \alpha q$ for $\alpha \in \mathbb{C} \setminus \{0\}$. A metric is needed in the quotient space $\mathbb{C}[x_1, \ldots, x_\ell]/\sim$ but not seen in the literature. We propose a scaling-invariant distance between polynomials p and q as the sine of the principal angle between the subspaces $span\{[\![p]\!]\}$ and $span\{[\![q]\!]\}$, denoted by

$$
\sin(p, q) \quad := \quad \begin{cases} 0 & \text{if} \quad p = q = 0 \\ 1 & \text{if} \quad p = 0,\ q \neq 0 \text{ or } p \neq 0,\ q = 0 \\ \left\| \frac{p}{\|p\|} - \frac{[\![q]\!] \cdot [\![p]\!]}{\|q\|\,\|p\|} \frac{q}{\|q\|} \right\| & \text{if} \quad p \neq 0,\ q \neq 0. \end{cases} \tag{2}
$$

Here the "$\cdot$" denotes the standard vector dot product. Let P_f be the projection mappings to $span\{[\![f]\!]\}$ for any polynomial f. It is known that $\sin(p, q) \equiv \|P_p - P_q\|_2$ (c.f. [28]), and is thus a distance in the quotient space $\mathbb{C}[x_1, \ldots, x_l]/\sim$.

A polynomial f is **factorable** if there exist nonconstant polynomials g and h such that $f = g\,h$, otherwise it is **irreducible**. We say $\alpha f_1\, f_2 \cdots f_k$ is a **factorization** of f if $\alpha \in \mathbb{C}$ is a constant scalar and the factors $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_l]$ so that the product $\alpha f_1 \cdots f_k$ is equivalent to f. Here we abuse the notation $\alpha f_1 \cdots f_k$ as it represents either the polynomial product or the factorization that consists of the array of factors $\alpha, f_1, \ldots, f_k$, depending on the context.

We say two factorizations $\alpha\, f_1\, f_2\, \cdots\, f_k$ and $\beta\, g_1\, g_2\, \cdots\, g_k$ of the same number of factors are equivalent, denoted by $\alpha\, f_1\, f_2\, \cdots\, f_k \sim \beta\, g_1\, g_2\, \cdots\, g_k$, if there is a permutation $\{\sigma_1, \ldots, \sigma_k\}$ of $\{1, \ldots, k\}$ such that $f_j \sim g_{\sigma_j}$ for $j = 1, \ldots, k$. For two factorizations $\alpha\, f_1\, f_2\, \cdots\, f_k$ and $\beta\, g_1\, g_2\, \cdots\, g_m$ with different number of factors, say $k < m$, they are equivalent if we can append constant factors $f_{k+1} = \cdots = f_m = 1$ so that $\alpha\, f_1\, f_2\, \cdots\, f_m \sim \beta\, g_1\, g_2\, \cdots\, g_m$. If $f_1, f_2, \cdots, f_k$ are all irreducible, then $\alpha f_1\, f_2\, \cdots\, f_k$ is an **irreducible factorization**. The irreducible factorization of a polynomial is unique as an equivalence class.

A factorization $\gamma g_1 \cdots g_m$ is regarded as an approximate factorization of f if the **backward error** $\sin(f, \gamma g_1 \cdots g_m)$ is small enough and acceptable in the underlying application. The **forward error** of the factorization $\gamma g_1 \cdots g_m$ is the difference between the factors $g_1, \ldots, g_m$ and their counterparts in $f = \alpha f_1 \cdots f_k$ via a proper metric that is needed but not properly established in the literature. Here we extend the distance measurement $\sin(\cdot, \cdot)$ to the **distance between two factorizations** as

$$
dist\big(\alpha f_1 \cdots f_k,\ \gamma g_1 \cdots g_k\big) \quad := \quad \min_{(\sigma_1, \ldots, \sigma_k) \in \Sigma} \Big\{ \max_{1 \leq j \leq k} \big\{ \sin(f_j, g_{\sigma_j}) \big\} \Big\} \tag{3}
$$

where Σ is the collection of all permutations $(\sigma_1, \ldots, \sigma_k)$ of $(1, \ldots, k)$. This distance extends to two factorizations with different numbers of factors by appending

dummy constant factors. Clearly, two factorizations are equivalent if and only if their distance is zero.

A polynomial is **squarefree** if its irreducible factorization consists of pairwise coprime factors. A **squarefree factorization** $\alpha f_1^{k_1} \cdots f_r^{k_r}$ consists of squarefree polynomials $f_1, \ldots, f_r$ as components that are pairwise coprime but may or may not be irreducible. Again, we use the notation $\alpha f_1^{k_1} f_2^{k_2} \cdots f_r^{k_r}$ to represent either the polynomial that equals to the result of the polynomial multiplication or the factorization consists of the factors $\alpha, \ f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_r, \ldots, f_r$ where each f_j repeats k_j times for $j = 1, \ldots, r$.

If $\alpha f_1^{k_1} \cdots f_r^{k_r}$ is an irreducible squarefree factorization of a polynomial f with degree $\mathbf{m}$, we shall use $\mathfrak{M} = \mathbf{m}_1^{k_1} \cdots \mathbf{m}_r^{k_r}$ to denote the **factorization structure**, or simply the **structure** of f, where $\mathbf{m}_j = deg(f_j) \neq \mathbf{0}$, $k_j \geq 1$ for $j = 1, \ldots, r$ and $k_1 \mathbf{m_1} + \cdots + k_r \mathbf{m_r} = \mathbf{m} = deg(f)$. We shall also say such an $\mathfrak{M}$ is one of the factorization structures of the degree $\mathbf{m}$ and denote $deg(\mathfrak{M}) = \mathbf{m}$. Any permutation of $\mathbf{m}_1^{k_1}, \ldots, \mathbf{m}_r^{k_r}$ in $\mathfrak{M} = \mathbf{m}_1^{k_1} \cdots \mathbf{m}_r^{k_r}$ is considered the same structure.

There are two cases for a factorization structure $\mathfrak{M}$ to be called **trivial** when $\mathfrak{M}$ is the factorization structure of either an irreducible polynomial or a univariate polynomial with no multiple roots. A factorization structure is **nontrivial** if it is not trivial.

3 Factorization Manifolds

The factorization of a polynomial f is an equivalence class in which a specific representative $\alpha f_1^{k_1} \cdots f_r^{k_r}$ can be extracted using a set of auxiliary equations

$$\mathbf{b}_1 \cdot [\![f_1]\!] \ = \ \cdots \ = \ \mathbf{b}_r \cdot [\![f_r]\!] \ = \ 1$$

where $\mathbf{b}_1, \ldots, \mathbf{b}_r$ are unit vectors of proper dimensions. We call such vectors $\mathbf{b}_1, \ldots, \mathbf{b}_r$ the **scaling vectors**. Scaling vectors can be chosen randomly. A more natural choice during computation is the normalized initial approximation of $[\![f_i]\!]$ so that $\mathbf{b}_i \cdot [\![f_i]\!] \approx \|f_i\|^2 = 1$ for $i = 1, \ldots, r$. For any factorizations $\gamma p_1^{k_1} \cdots p_r^{k_r}$ and $\mu q_1^{k_1} \cdots q_r^{k_r}$ scaled by equations $\mathbf{b}_i \cdot [\![p_i]\!] = \mathbf{b}_i \cdot [\![q_i]\!] = 1$ for $i = 1, \ldots, r$, it is clear that $\|p_1\|, \ldots, \|p_r\|, \|q_1\|, \ldots, \|q_r\| \geq 1$ since the scaling vectors are of unit norms, and the following lemma applies.

Lemma 1 *Let* $\gamma p_1^{k_1} \cdots p_r^{k_r}$ *and* $\mu q_1^{k_1} \cdots q_r^{k_r}$ *be two factorizations with components satisfying* $\|p_i\|, \|q_i\| \geq 1$ *for* $i = 1, \ldots, r$. *Then*

$$dist\big(\gamma p_1^{k_1} \cdots p_r^{k_r}, \, \mu q_1^{k_1} \cdots q_r^{k_r}\big) \quad \leq \quad \max_{1 \leq i \leq r} \big\| p_i - q_i \big\|. \tag{4}$$

PROOF. It is straightforward to verify that $\sin(p_i, q_i) \leq \|p_i - q_i\|$ whenever $\|p_i\|, \|q_i\| \geq 1$ for $i = 1, \ldots, r$. Thus (4) holds. $\qquad\square$

Suppose f possesses an irreducible squarefree factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$ and the degrees $deg(f_i) = \mathbf{m}_i$ for $i = 1, \ldots, r$. The factorization structure $\mathfrak{M}$ equals to $\mathbf{m}_1^{k_1} \cdots \mathbf{m}_r^{k_r}$. All the polynomials sharing this factorization structure form a subset

$$\mathscr{F}_{\mathfrak{M}} \ := \ \big\{ f \in \mathscr{R}_{\mathbf{m}} \ \big| \ f = \alpha g_1^{k_1} \cdots g_r^{k_r} \text{ where } \alpha \in \mathbb{C}, \ g_j \in \mathscr{R}_{\mathbf{m}_j}, \ j = 1, \ldots, r$$
$$\text{are irreducible and pairwise coprime}\big\}$$

of $\mathcal{R}_{\mathbf{m}}$ where $\mathbf{m} = deg(\mathfrak{M})$. For almost all unit scaling vectors $\mathbf{b}_i \in \mathbb{C}^{\langle \mathbf{m}_i \rangle}$ for $i = 1, \ldots, r$, a polynomial $f \in \mathscr{F}_{\mathfrak{M}}$ possesses irreducible factors $\alpha, f_1, \ldots, f_r$ such that $\alpha f_1^{k_1} \cdots f_r^{k_r} = f$ and $\mathbf{b}_1 \cdot [\![f_1]\!] = \cdots = \mathbf{b}_r \cdot [\![f_r]\!] = 1$ so that the array $\big(\gamma, [\![p_1]\!], \ldots, [\![p_r]\!] \big) = \big(\alpha, [\![f_1]\!], \ldots, [\![f_r]\!] \big)$ is a solution to the equation

$$\phi \big(\gamma, [\![p_1]\!], \ldots, [\![p_r]\!] \big) \;=\; \big([\![f]\!], 1, \ldots, 1 \big) \tag{5}$$

for $\gamma \in \mathbb{C}$, $p_j \in \mathcal{R}_{\mathbf{m}_j}$, $j = 1, \ldots, r$, where the mapping ϕ is defined by

$$\begin{aligned} \phi \;:\; \mathbb{C} \times \mathbb{C}^{\langle \mathbf{m}_1 \rangle} \times \cdots \times \mathbb{C}^{\langle \mathbf{m}_r \rangle} \;&\longrightarrow\; \mathbb{C}^{\langle \mathbf{m} \rangle} \times \mathbb{C} \times \cdots \times \mathbb{C} \\ \big(\gamma, [\![p_1]\!], \ldots, [\![p_r]\!] \big) \;&\longmapsto\; \big([\![\gamma p_1^{k_1} \cdots p_r^{k_r}]\!], \, \mathbf{b}_1 \cdot [\![p_1]\!], \, \ldots, \, \mathbf{b}_r \cdot [\![p_r]\!] \big). \end{aligned} \tag{6}$$

Let $q_i = \left(\frac{k_i}{p_i} \right) \alpha p_1^{k_1} \cdots p_r^{k_r}$. Then the Jacobian of ϕ can be written as

$$\mathcal{J} \big(\alpha, [\![p_1]\!], \ldots, [\![p_r]\!] \big) \;=$$
$$\begin{bmatrix} column([\![p_1^{k_1} \cdots p_r^{k_r}]\!]) & C_{\mathbf{m}_1}(q_1) & C_{\mathbf{m}_2}(q_2) & \cdots & C_{\mathbf{m}_r}(q_r) \\ & column(\mathbf{b}_1)^{\mathsf{H}} & & & \\ & & column(\mathbf{b}_2)^{\mathsf{H}} & & \\ & & & \ddots & \\ & & & & column(\mathbf{b}_r)^{\mathsf{H}} \end{bmatrix} \tag{7}$$

where $column(\cdot)$ represents the column block generated by a vector $(\cdot)$, the notation $(\cdot)^{\mathsf{H}}$ denotes the Hermitian transpose of the matrix $(\cdot)$, and $C_{\mathbf{m}_i}(q_i)$ is the convolution matrix [30] associated with q_i so that $C_{\mathbf{m}_i}(q_i) \cdot [\![h]\!] = [\![q_i h]\!]$ holds for any $h \in \mathcal{R}_{\mathbf{m}_i}$, $i = 1, \ldots, r$. We need several lemmas for establishing the main theorems of the paper.

Lemma 2 *For* $\alpha \in \mathbb{C} \setminus \{0\}$, $\mathbf{b}_j \in \mathbb{C}^{\langle \mathbf{m}_j \rangle}$ *and* $p_j \in \mathscr{R}_{\mathbf{m}_j}$ *with* $\mathbf{b}_j \cdot [\![p_j]\!] \neq 0$ *for* $j = 1, \ldots, r$, *the Jacobian in* (7) *is injective if and only if* $p_1, \ldots, p_r$ *are pairwise coprime.*

PROOF. Assume $p_1, \ldots, p_r$ are pairwise coprime and the matrix-vector multiplication
$$\mathcal{J} \big(\alpha, [\![p_1]\!], \ldots, [\![p_r]\!] \big) \cdot \big(-a, [\![v_1]\!], \ldots, [\![v_r]\!] \big) \;=\; \mathbf{0}. \tag{8}$$
Then $\mathbf{b}_1 \cdot [\![v_1]\!] = \cdots = \mathbf{b}_r \cdot [\![v_r]\!] = 0$ as well as $\sum_{i=1}^{r} q_i v_i = a \prod_{i=1}^{r} p_i^{k_i}$ that lead to $\sum_{i=1}^{r} k_i \alpha p_1 \cdots p_{i-1} v_i p_{i+1} \cdots p_r = a p_1 p_2 \cdots p_r$. Thus

$$k_1 \, \alpha \, v_1 \, p_2 \, \cdots \, p_r \;=\; a \, p_1 \, p_2 \, \cdots \, p_r - \sum_{i=2}^{r} k_i \, \alpha \, p_1 \, \cdots \, p_{i-1} \, v_i \, p_{i+1} \, \cdots \, p_r$$

that contains the factor p_1. Because $gcd(p_1, p_j) = 1$ for $j = 2, \ldots, r$, there is a polynomial s such that $v_1 = s p_1$. The degree $deg(v_1) \leq deg(p_1)$ leads to s being a constant. Since $\mathbf{b}_1 \cdot [\![p_1]\!] \neq 0$, $0 = \mathbf{b}_1 \cdot [\![v_1]\!] = s \, \mathbf{b}_1 \cdot [\![p_1]\!]$, hence $s = 0$. Consequently $v_1 = 0$. Similarly we can prove that $v_i = 0$ for $i = 2, \ldots, r$. Substituting $v_1 = \cdots = v_r = 0$ into (8), we have $a p_1^{k_1} \cdots p_r^{k_r} = 0$ and thus $a = 0$. Therefore, the Jacobian is injective.

Conversely, to prove that the injectivity of the Jacobian in (7) implies $p_1, \ldots, p_r$ are pairwise coprime, assume there are some $i \neq j$ such that $gcd(p_i, p_j) \neq 1$. Then we shall prove that the Jacobian must be rank-deficient. Without loss of

generality, we can assume $p_1 = e\,s$ and $p_2 = e\,t$ for some polynomials e, s and t where $e = \gcd(p_1, p_2)$ is nonconstant. Then there are three possible cases. As case one, if $\mathbf{b}_1 \cdot [\![s]\!] = 0$ and $\mathbf{b}_2 \cdot [\![t]\!] = 0$, then it is easy to show that $\left(0, \frac{1}{k_1}[\![s]\!], \frac{-1}{k_2}[\![t]\!], \mathbf{0}, \ldots, \mathbf{0}\right)$ is a nonzero solution to (8). As case two, if $\mathbf{b}_1 \cdot [\![s]\!] = c \neq 0$ and $\mathbf{b}_2 \cdot [\![t]\!] = 0$, then we can consider $w = \frac{c}{\beta_1}\, e\, s - s$, where $\beta_1 = \mathbf{b}_1 \cdot [\![p_1]\!]$. It is straightforward to verify that

$$\mathbf{b}_1 \cdot [\![w]\!] \;\; = \;\; \frac{c}{\beta_1}\, \mathbf{b}_1 \cdot [\![p_1]\!] - \mathbf{b}_1 \cdot [\![s]\!] \;\; = \;\; c - c \;\; = \;\; 0.$$

Since $e = \gcd(p_1, p_2)$ which is nontrivial, we have $\deg(ces) > \deg(s)$ and consequently $w \neq 0$. Thus $\left(\frac{-c}{\beta_1}\alpha, \frac{1}{k_1}[\![w]\!], \frac{1}{k_2}[\![t]\!], \mathbf{0}, \ldots, \mathbf{0}\right)$ is a nonzero solution of (8). For the third case where $\mathbf{b}_1 \cdot [\![s]\!] = c \neq 0$ and $\mathbf{b}_2 \cdot [\![t]\!] = d \neq 0$, let $v_1 = \frac{c}{\beta_1}\, e\, s - s$ and $v_2 = -\frac{d}{\beta_2}\, e\, t + t$ where $\beta_2 = \mathbf{b}_2 \cdot [\![p_2]\!]$. Then $\left(\frac{d\alpha}{\beta_2} - \frac{c\alpha}{\beta_1}, \frac{1}{k_1}[\![v_1]\!], \frac{1}{k_2}[\![v_2]\!], \mathbf{0}, \ldots, \mathbf{0}\right)$ is a nonzero solution of (8). Therefore, the Jacobian is a rank-deficient matrix. $\quad\square$

Lemma 3 *Let* $\mathfrak{M} = \mathbf{m}_1^{k_1} \cdots \mathbf{m}_r^{k_r}$ *be a factorization structure of degree* $\mathbf{m}$ *and assume a sequence* $\{p_j\}_{j=1}^{\infty} \subset \mathscr{F}_{\mathfrak{M}}$ *converges to* $q \in \mathscr{R}_{\mathbf{m}}$. *Then there is a subsequence of* $\{p_j\}_{j=1}^{\infty}$ *whose irreducible factorizations converge to a factorization* $\alpha q_1^{k_1} \cdots q_r^{k_r}$ *of* q *with* $\deg(q_i) = \mathbf{m}_i$ *for* $i = 1, \ldots, r$. *Further assume* $q \in \mathscr{F}_{\mathfrak{M}}$. *Then the irreducible factorizations of* $\{p_j\}_{j=1}^{\infty}$ *converge to the irreducible factorization of* q.

PROOF. Let $\mathfrak{S}_i = \left\{ [\![f]\!] \;\middle|\; f \in \mathcal{R}_{\mathbf{m_i}}, \, \|f\| = 1 \right\}$ and $p_j = \alpha_j p_{j1}^{k_1} \cdots p_{jr}^{k_r}$ be an irreducible factorization of p_j, where $p_{ji} \in \mathfrak{S}_i$ for $i \in \{1, \ldots, r\}$ and $j = 1, 2, \ldots$. Denote

$$\mathbf{P}_j \;\; = \;\; (\alpha_j, [\![p_{j1}]\!], \ldots, [\![p_{jr}]\!]) \;\; \in \;\; \mathbb{C} \times \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r.$$

There is a subsequence $\{j_1, j_2, \ldots\}$ of $\{1, 2, \ldots\}$ such that $\lim_{\sigma \to \infty} \|p_{j_\sigma i} - q_i\| = 0$ for $i \in \{1, \ldots, r\}$ since $\mathfrak{S}_i$'s are compact. As a result, the subsequence $\{\alpha_{j_\sigma}\}$ converges to certain $\alpha \in \mathbb{C}$. Namely, the subsequence $\{\mathbf{P}_{j_\sigma}\}_{\sigma=1}^{\infty}$ converges to a point $(\alpha, [\![q_1]\!], \ldots, [\![q_r]\!])$ such that $q = \alpha q_1^{k_1} \cdots q_r^{k_r}$ and $\deg(q_i) \leq \mathbf{m}_i$ for $i = 1, \ldots, r$. From $\deg(q) = \mathbf{m}$ we have $\deg(q_i) = \mathbf{m}_i$ for $i = 1, \ldots, r$. By Lemma 1, the irreducible factorizations of p_{j_σ} for $\sigma = 1, 2, \ldots$ converge to the factorization $\alpha q_1^{k_1} \cdots q_r^{k_r}$ since

$$\mathit{dist}(\alpha_{j_\sigma} p_{j_\sigma 1}^{k_1} \cdots p_{j_\sigma r}^{k_r}, \; \alpha q_1^{k_1} \cdots q_r^{k_r}) \;\; \leq \;\; \max_i \|p_{j_\sigma i} - q_i\| \;\; \longrightarrow \;\; 0$$

when $\sigma \to \infty$. Moreover, if $q \in \mathscr{F}_{\mathfrak{M}}$, then $\alpha q_1^{k_1} \cdots q_r^{k_r}$ is an irreducible squarefree factorization of q by the uniqueness of factorizations. Furthermore, the irreducible squarefree factorizations of the whole sequence $\{p_j\}$ must converge to the factorization $\alpha q_1^{k_1} \cdots q_r^{k_r}$ since otherwise there would be a $\delta > 0$ and a subsequence of $\{\mathbf{P}_j\}_{j=1}^{\infty}$ converging to $(\hat{\alpha}, \hat{q}_1, \ldots, \hat{q}_r)$, with $q = \hat{\alpha} \hat{q}_1^{k_1} \cdots \hat{q}_r^{k_r}$ and $\mathit{dist}\left(\hat{\alpha} \hat{q}_1^{k_1} \cdots \hat{q}_r^{k_r}, \alpha q_1^{k_1} \cdots q_r^{k_r}\right) \geq \delta$, contradicting the uniqueness of the factorization of q. $\quad\square$

Lemma 3 directly leads to the following corollaries.

Corollary 1 *Let* f *be a polynomial with a factorization structure* $\mathfrak{M} = \mathbf{m}_1^{k_1} \cdots \mathbf{m}_r^{k_r}$ *of degree* $\mathbf{m}$ *and an irreducible squarefree factorization* $\alpha f_1^{k_1} \cdots f_r^{k_r}$ *whose components satisfies* $\|f_1\| = \cdots = \|f_r\| = 1$. *For any* $\epsilon > 0$, *there is a neighborhood* Ω_f

of f in $\mathcal{R}_{\mathbf{m}}$ such that every $g \in \Omega_f \cap \mathscr{F}_{\mathfrak{M}}$ corresponds to a unique $(\beta, g_1, \ldots, g_r)$ with $g = \beta g_1^{k_1} \cdots g_r^{k_r}$, $[\![f_1]\!] \cdot [\![g_1]\!] = \cdots = [\![f_r]\!] \cdot [\![g_r]\!] = 1$ and

$$\sqrt{|\alpha - \beta|^2 + \|f_1 - g_1\|^2 + \cdots + \|f_r - g_r\|^2} \quad < \quad \epsilon.$$

PROOF. For any $\delta > 0$, Lemma 3 implies that there is a neighborhood $\Omega_{f,\delta}$ of f in $\mathcal{R}_{\mathbf{m}}$ such that the irreducible squarefree factorization $\beta g_1^{k_1} \cdots g_r^{k_r}$ of every $g \in \Omega_{f,\delta} \cap \mathscr{F}_{\mathfrak{M}}$ satisfies $dist\big(\beta g_1^{k_1} \cdots g_r^{k_r}, \alpha f_1^{k_1} \cdots f_r^{k_r}\big) < \delta$. We can assume $\delta < \frac{1}{2} \min_{i \neq j}\{\sin(f_i, f_j)\}$ and $\max_j \{\sin(f_j, g_j)\} < \delta$. Since $i \neq j$ implies

$$\sin(f_i, g_j) \quad \geq \quad \sin(f_i, f_j) - \sin(f_j, g_j) \quad > \quad \delta,$$

no the other permutations of $g_1, \ldots, g_r$ satisfy $\max_j\{\sin(f_j, g_j)\} < \delta$. Further assume $g_1, \ldots, g_r$ are the unique representatives in their respective equivalence classes satisfying $[\![f_j]\!] \cdot [\![g_j]\!] = 1$ for $j = 1, \ldots, r$. Then

$$[\![f_j]\!] \cdot [\![f_j - g_j]\!] \quad = \quad 0, \quad \|f_j - g_j\| = \|g_j\| \sin(f_j, g_j) \quad \text{and}$$
$$\|g_j\|^2 \quad = \quad \|f_j\|^2 + \|f_j - g_j\|^2 \quad = \quad 1 + \|g_j\|^2 \sin^2(f_j, g_j),$$

leading to

$$\|f_j - g_j\| \quad = \quad \frac{\sin(f_j, g_j)}{\sqrt{1 - \sin^2(f_j, g_j)}} \quad < \quad \frac{\delta}{\sqrt{1 - \delta^2}}$$

for $j = 1, \ldots, r$. Therefore, for any $\epsilon > 0$, the assertion holds when δ is small. $\square$

Corollary 2 *Polynomials of degree $\mathbf{m}$ with a trivial factorization structure form an open subset of $\mathcal{R}_{\mathbf{m}}$.*

PROOF. For a univariate degree $\mathbf{m}$, the assertion follows from the continuity of polynomial roots with respect to the coefficients. Assume $\mathbf{m}$ is multivariate and the assertion does not hold. Then there is an irreducible polynomial f of degree $\mathbf{m}$ and a sequence of factorable polynomials $\{p_j\}_{j=1}^{\infty}$ approaching f. Because there are finitely many factorization structures in $\mathcal{R}_{\mathbf{m}}$, there exists a nontrivial factorization structure $\mathfrak{M}$ and a subsequence $\{p_{j_\sigma}\}_{\sigma=1}^{\infty}$ in $\mathscr{F}_{\mathfrak{M}}$. By Lemma 3, the irreducible factorizations of this subsequence converge to a nontrivial factorization of f, contradicting the irreducibility of f. $\square$

We can now establish the following Factorization Manifold Theorem. A subset S in the topological space $\mathcal{R}_{\mathbf{m}}$ is a **complex analytic manifold** of dimension k in $\mathcal{R}_{\mathbf{m}}$ if, for every $p \in S$, there exists an open subset Ω of $\mathcal{R}_{\mathbf{m}}$ containing p and a biholomorphic mapping from $[\![S \cap \Omega]\!] \subset \mathbb{C}^{\langle \mathbf{m} \rangle}$ onto an open subset of $\mathbb{C}^k$. The **codimension**, namely the dimension deficit, of S is denoted by $codim(S) := dim(\mathcal{R}_{\mathbf{m}}) - dim(S) = \langle \mathbf{m} \rangle - k$. The Factorization Manifold Theorem is at core of the geometry on the polynomial factorization. This result and the proof are fundamental but not seen in the literature.

Theorem 3 (Factorization Manifold Theorem) *Let $\mathfrak{M} = \mathbf{m}_1^{k_1} \cdots \mathbf{m}_r^{k_r}$ be a factorization structure with $deg\big(\mathfrak{M}\big) = \mathbf{m}$. Then $\mathscr{F}_{\mathfrak{M}}$ is a complex analytic manifold in $\mathcal{R}_{\mathbf{d}}$ and*

$$codim(\mathscr{F}_{\mathfrak{M}}) \quad = \quad \langle \mathbf{m} \rangle - \Big(\langle \mathbf{m}_1 \rangle + \cdots + \langle \mathbf{m}_r \rangle + 1 - r \Big). \tag{9}$$

PROOF. Let $f \in \mathscr{F}_{\mathfrak{M}}$ with an irreducible squarefree factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$ along with $deg(f_j) = \mathbf{m}_j$, $\|f_j\| = 1$ and $\mathbf{b}_j = [\![f_j]\!]$ for $j = 1, \ldots, r$ so that (6) defines a holomorphic mapping $\phi : \mathbb{C}^k \to \mathbb{C}^{\langle \mathbf{m} \rangle + r}$ with $k = 1 + \langle \mathbf{m}_1 \rangle + \cdots + \langle \mathbf{m}_r \rangle$ and

$$\phi\big(\alpha, [\![f_1]\!], \ldots, [\![f_r]\!]\big) = \big([\![f]\!], 1, \ldots, 1\big).$$

By Corollary 2, there is a neighborhood Δ of $\big(\alpha, [\![f_1]\!], \ldots, [\![f_r]\!]\big)$ in the space $\mathbb{C} \times \mathbb{C}^{\langle \mathbf{m}_1 \rangle} \times \cdots \times \mathbb{C}^{\langle \mathbf{m}_r \rangle}$ and every $\big(\tilde{\alpha}, [\![\tilde{f}_1]\!], \ldots, [\![\tilde{f}_r]\!]\big) \in \Delta$ forms an irreducible squarefree factorization $\tilde{\alpha} \tilde{f}_1^{k_1} \cdots \tilde{f}_r^{k_r}$. By Lemma 2, the Jacobian of ϕ is of full rank k at $\big(\alpha, [\![f_1]\!], \ldots, [\![f_r]\!]\big)$. As a result, the Inverse Mapping Theorem ensures that certain k components of ϕ form a biholomorphic mapping $\check{\phi}$ from an open neighborhood Σ of $\big(\alpha, [\![f_1]\!], \ldots, [\![f_r]\!]\big)$ in $\mathbb{C}^k$ to an open subset Π of $\mathbb{C}^k$. We can assume $\Sigma \subset \Delta$.

The mapping $\check{\phi}$ must contain the last r components of ϕ since $\check{\phi}$ would not be injective without those scaling constraints. Without loss of generality, we assume $\check{\phi}$ consists of the last k components of ϕ and we split $\phi(\mathbf{x})$ into $\hat{\phi}(\mathbf{x}) = \mathbf{u}$, and $\check{\phi}(\mathbf{x}) = (\mathbf{v}, \mathbf{w})$ where $\mathbf{u} \in \mathbb{C}^{\langle \mathbf{m} \rangle + r - k}$, $\mathbf{v} \in \mathbb{C}^{k-r}$ and $\mathbf{w} \in \mathbb{C}^r$. Let

$$\check{\Pi} = \left\{ \mathbf{v} \in \mathbb{C}^{k-r} \mid (\mathbf{v}, 1, \ldots, 1) \in \Pi \right\}$$

which is open in $\mathbb{C}^{k-r}$. Then the holomorphic mapping

$$\begin{aligned} \mu : \quad \check{\Pi} &\longrightarrow \quad \mathbb{C}^{\langle \mathbf{m} \rangle} \\ \mathbf{v} &\longmapsto \quad \big(\hat{\phi} \circ \check{\phi}^{-1}(\mathbf{v}, 1, \ldots, 1), \mathbf{v}\big) \end{aligned}$$

maps $\tilde{\Pi}$ to $\mu(\tilde{\Pi}) \subset \mathscr{F}_{\mathfrak{M}}$ since $\tilde{\Pi} \times \{(1, \ldots, 1)\} \subset \check{\Pi}$ and $\Sigma \subset \Delta$. Define the projection

$$\begin{aligned} \psi : \quad \mathbb{C}^{\langle \mathbf{m} \rangle + r - k} \times \check{\Pi} &\longrightarrow \quad \check{\Pi} \\ (\mathbf{u}, \mathbf{v}) &\longmapsto \quad \mathbf{v}. \end{aligned}$$

By Corollary 1, there is an open neighborhood $\Omega \subset \mathbb{C}^{\langle \mathbf{m} \rangle - k + r} \times \check{\Pi}$ of $[\![f]\!]$ in $\mathbb{C}^{\langle \mathbf{m} \rangle}$ such that every $[\![p]\!] \in \Omega \cap [\![\mathscr{F}_{\mathfrak{M}}]\!]$ corresponds to a unique $(\gamma, [\![p_1]\!], \ldots, [\![p_r]\!]) \in \Sigma$ with $\phi(\gamma, [\![p_1]\!], \ldots, [\![p_r]\!]) = ([\![p]\!], 1, \ldots, 1)$, namely $\mu \circ \psi([\![p]\!]) = [\![p]\!]$. Define

$$\tilde{\Pi} = \psi(\Omega \cap [\![\mathscr{F}_{\mathfrak{M}}]\!]).$$

We have $\mu(\tilde{\Pi}) \supset \Omega \cap [\![\mathscr{F}_{\mathfrak{M}}]\!])$. Then for every $\mathbf{v} \in \tilde{\Pi}$, there is a $\mathbf{u} \in \mathbb{C}^{\langle \mathbf{m} \rangle + r - k}$ such that $(\mathbf{u}, \mathbf{v}) \in \Omega \cap [\![\mathscr{F}_{\mathfrak{M}}]\!]$ corresponds to a unique $(\gamma, [\![p_1]\!], \ldots, [\![p_r]\!]) \in \Sigma$ with $\phi(\gamma, [\![p_1]\!], \ldots, [\![p_r]\!]) = (\mathbf{u}, \mathbf{v}, 1, \ldots, 1)$, implying

$$\mathbf{u} = \hat{\phi} \circ \check{\phi}^{-1}(\mathbf{v}, 1, \ldots, 1) \quad \text{and} \quad \mu(\mathbf{v}) = (\mathbf{u}, \mathbf{v}).$$

Namely $\mu(\tilde{\Pi}) \subset \Omega \cap [\![\mathscr{F}_{\mathfrak{M}}]\!]$ and thus $\mu(\tilde{\Pi}) = \Omega \cap [\![\mathscr{F}_{\mathfrak{M}}]\!]$. Since Ω is open and μ is continuous, hence $\mu^{-1}(\Omega) = \mu^{-1}(\Omega \cap \mathscr{F}_{\mathfrak{M}}) = \tilde{\Pi}$ is open in $\mathbb{C}^{k-r}$. Therefore, ψ is biholomorphic from $\Omega \cap [\![\mathscr{F}_{\mathfrak{M}}]\!]$ onto $\tilde{\Pi}$ with the inverse μ. Namely $\mathscr{F}_{\mathfrak{M}}$ is a complex analytic manifold of dimension $k - r$, and (9) follows. $\square$

We shall refer to $\mathscr{F}_{\mathfrak{M}}$ as the **factorization manifold** associated with the factorization structure $\mathfrak{M}$. Its dimension deficit indicates how ill-posed the factorization is for polynomials on the manifold. For a polynomial p of degree $\mathbf{m}$, we say the **singularity** of p and its factorization structure $\mathfrak{M}$ is k if $\mathscr{F}_{\mathfrak{M}}$ is of codimension k in $\mathcal{R}_{\mathbf{m}}$. A polynomial is **singular** in terms of factorization if its singularity is positive, or **nonsingular** otherwise.

Corollary 4 *A polynomial is singular if and only if its factorization structure is nontrivial, and nonsingular polynomials of degree* **m** *form an open dense subset of* $\mathcal{R}_{\mathbf{m}}$.

PROOF. For both type of trivial factorization structures, the corresponding factorization manifold has a singularity zero from (9) by a straightforward verification. To prove $\mathit{codim}(\mathscr{F}_{\mathfrak{M}}) > 0$ for any nontrivial structure $\mathfrak{M}$, it suffices to show that for any degrees $\hat{\mathbf{n}} \neq \mathbf{0}$ and $\check{\mathbf{n}} \neq \mathbf{0}$ such that $\hat{\mathbf{n}} + \check{\mathbf{n}}$ is a non-univariate degree, we have

$$\langle \hat{\mathbf{n}} + \check{\mathbf{n}} \rangle \quad > \quad \langle \hat{\mathbf{n}} \rangle + \langle \check{\mathbf{n}} \rangle - 1. \tag{10}$$

In fact, if is straightforward to verify (10) for $\ell = 2$, namely we have the inequality

$$(\hat{n}_1 + \check{n}_1 + 1)(\hat{n}_2 + \check{n}_2 + 1) \quad > \quad (\hat{n}_1 + 1)(\hat{n}_2 + 1) + (\check{n}_1 + 1)(\check{n}_2 + 1) - 1$$

if $\hat{n}_1 + \check{n}_1 > 0$ and $\hat{n}_2 + \check{n}_2 > 0$, and the inequality (10) for any positive integer $\ell \geq 2$ follows an induction.

Let $\mathfrak{M}$ be trivial. $\mathscr{F}_{\mathfrak{M}}$ is open in $\mathcal{R}_{\mathbf{m}}$ by Corollary 2. It is dense in $\mathcal{R}_{\mathbf{m}}$ since it equals $\mathcal{R}_{\mathbf{m}}$ minus finitely many singular factorization manifolds of lower dimensions. □

Corollary 4 provides an ultimate explanation why polynomial factorization is an ill-posed problem: Any polynomial p having a nontrivial factorization is singular in terms of factorization. Almost any perturbation Δp results in the data $\tilde{p} = p + \Delta p$ that is pushed off the native manifold into the open dense subset of nonsingular polynomials, altering the factorization to a trivial one. This discontinuity makes the conventional factorization ill-posed and a challenge in numerical computation. When the factorization structure is preserved, however, the irreducible factorization is Lipschitz continuous as asserted in the following corollary. It is this continuity that makes numerical factorization possible.

Corollary 5 (Factorization Continuity Theorem) The irreducible squarefree factorization is locally Lipschitz continuous on a factorization manifold: *For any polynomial* $f \in \mathscr{F}_{\mathfrak{M}}$ *with an irreducible squarefree factorization* $\alpha f_1^{k_1} \cdots f_r^{k_r}$, *there are constants* $\delta, \eta > 0$ *such that, for every polynomial* $g \in \mathscr{F}_{\mathfrak{M}}$ *satisfying* $\|f - g\| < \delta$, *the irreducible squarefree factorization* $\beta g_1^{k_1} \cdots g_r^{k_r}$ *of* g *satisfies*

$$\mathit{dist}\big(\alpha f_1^{k_1} \cdots f_r^{k_r}, \ \beta g_1^{k_1} \cdots g_r^{k_r}\big) \quad \leq \quad \eta \, \|f - g\|.$$

PROOF. We can assume $\|f_1\| = \cdots = \|f_r\| = 1$ and thus define the mapping ϕ in (6) with $\mathbf{b}_j = [\![f_j]\!]$ for $j = 1, \ldots, r$. Using the notations in the proof of Theorem 3, the mapping $\zeta([\![g]\!]) = \check{\phi}^{-1}\big(\psi([\![g]\!]), 1, \ldots, 1\big)$ is holomorphic and thus Lipschitz continuous for any $g \in \mathscr{F}_{\mathfrak{M}}$ near f. Thus the assertion of this corollary follows from Lemma 1. □

4 Geometry of Factorization Manifolds

Factorization manifolds form a topologically stratified space $\mathcal{R}_{\mathbf{m}}$ in which every singular factorization manifold is embedded in manifolds of lower singularities as we shall elaborate in detail. There are two embedding operations on a factorization structure:

– The **degree combining** operation is adding two ℓ-tuple degrees of the same multiplicity while keeping other components of the factorization structure unchanged:

$$\cdots \mathbf{n}_i^{k_i} \cdots \mathbf{n}_j^{k_j} \cdots \quad \longrightarrow \quad \cdots (\mathbf{n}_i + \mathbf{n}_j)^k \cdots \quad \text{where} \quad k_i = k_j = k. \quad (11)$$

– The **multiplicity splitting** operation decomposes a component of a factorization structure into two as follows:

$$\cdots \mathbf{n}_i^{k_i} \cdots \quad \longrightarrow \quad \cdots \mathbf{n}_i^{\hat{k}_i} \, \mathbf{n}_i^{\tilde{k}_i} \cdots \quad \text{where} \quad k_i = \hat{k}_i + \tilde{k}_i. \quad (12)$$

A factorization structure $\mathfrak{N}$ is **embedded** in $\mathfrak{M}$, denoted by $\mathfrak{N} \prec \mathfrak{M}$, if $\mathfrak{N} = \mathfrak{M}$ or $\mathfrak{M}$ can be obtained by applying a sequence of embedding operations on $\mathfrak{N}$. For example,

$$(4,3)^5 (1,6)^2 (3,2) \prec (4,3)^3 (4,3)^2 (1,6)^2 (3,2) \quad (\text{splitting } (4,3)^5 \text{ to } (4,3)^3 (4,3)^2)$$
$$\prec (4,3)^3 (5,9)^2 (3,2) \qquad (\text{combining } (4,3)^2 (1,6)^2 \text{ to } (5,9)^2)$$

The relation $\prec$ is a partial ordering among factorization structures.

Theorem 6 (Factorization Manifold Embedding Theorem) *Let $\mathfrak{N}$ be a factorization structure and $f \in \mathscr{F}_{\mathfrak{N}}$. For any factorization structure $\mathfrak{M}$ with $deg(\mathfrak{N}) = deg(\mathfrak{M}) = \mathbf{m}$, we have $f \in \overline{\mathscr{F}_{\mathfrak{M}}}$ if and only if $\mathfrak{N} \prec \mathfrak{M}$. Furthermore, $codim(\mathscr{F}_{\mathfrak{N}}) > codim(\mathscr{F}_{\mathfrak{M}})$ in $\mathcal{R}_{\mathbf{m}}$ if $\mathfrak{N} \prec \mathfrak{M}$ and $\mathfrak{N} \neq \mathfrak{M}$.*

PROOF. Assume $\mathfrak{N} \prec \mathfrak{M}$. To prove $f \in \overline{\mathscr{F}_{\mathfrak{M}}}$, it suffices to show $f \in \overline{\mathscr{F}_{\tilde{\mathfrak{N}}}}$ if $\tilde{\mathfrak{N}}$ is obtained from $\mathfrak{N}$ by one of the embedding operations (11) and (12). If $\tilde{\mathfrak{N}}$ is obtained by degree combining (11), then we can write $f = f_i^k f_j^k g$ with $f_i \in \mathscr{R}_{\mathbf{n}_i}$ and $f_j \in \mathscr{R}_{\mathbf{n}_j}$ being irreducible and coprime. By Corollary 4, there is a polynomial sequence $\{h_l\}_{l=1}^{\infty} \subset \mathscr{R}_{\mathbf{n}_i + \mathbf{n}_j}$ converging to zero such that $f_i f_j + h_l$ is irreducible for all $l = 1, 2, \ldots$. Thus $f \in \overline{\mathscr{F}_{\tilde{\mathfrak{N}}}}$ since $(f_i f_j + h_l)^k g \in \mathscr{F}_{\tilde{\mathfrak{N}}}$ converges to f for $l \to \infty$. If $\tilde{\mathfrak{N}}$ is obtained by multiplicity splitting (12), then we can write $f = f_i^{k_i} g$ with $f_i \in \mathscr{R}_{\mathbf{n}_i}$. There is a sequence $\{h_l\}_{l=1}^{\infty} \subset \mathscr{R}_{\mathbf{n}_i}$ converging to zero such that $f_i + h_l$ is irreducible for all $l = 1, 2, \ldots$. Thus $f \in \overline{\mathscr{F}_{\tilde{\mathfrak{N}}}}$ since $f_i^{\hat{k}_i} (f_i + h_l)^{\tilde{k}_i} g \in \mathscr{F}_{\tilde{\mathfrak{N}}}$ with $\hat{k}_i + \tilde{k}_i = k_i$ converges to f for $l \to \infty$.

Conversely, assume $f \in \overline{\mathscr{F}_{\mathfrak{M}}}$ with an irreducible squarefree factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$. There is a sequence $\{g_l\}_{l=1}^{\infty} \subset \mathscr{F}_{\mathfrak{M}}$ converging to f. Write $\mathfrak{M} = \mathbf{m}_1^{k_1'} \cdots \mathbf{m}_s^{k_s'}$ and $g_l = \beta_l g_{l1}^{k_1'} \cdots g_{ls}^{k_s'}$ for $l = 1, 2, \ldots$. Clearly $deg(\mathfrak{M}) \geq deg(\mathfrak{N})$. If $deg(\mathfrak{M}) = deg(\mathfrak{N})$, then Lemma 3 implies that we can further assume the limit $\lim\limits_{l \to \infty} g_{lj} = \hat{g}_j \in \mathscr{R}_{\mathbf{m}_j}$ for $j = 1, \ldots, s$ and $\beta_l \to \hat{\beta}$. Due to $\alpha f_1^{k_1} \cdots f_r^{k_r} = \hat{\beta} \hat{g}_1^{k_1'} \cdots \hat{g}_s^{k_s'}$ and the uniqueness of factorizations, we can factor polynomials $\hat{g}_1, \ldots, \hat{g}_s$ and combine equivalent irreducible factors into higher multiplicities to reproduce the squarefree irreducible factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$. Namely, the structure $\mathfrak{M}$ can be obtained by a sequence of embedding operations on $\mathfrak{N}$, leading to $\mathfrak{N} \prec \mathfrak{M}$.

The inequality $codim(\mathscr{F}_{\mathfrak{N}}) > codim(\mathscr{F}_{\mathfrak{M}})$ follows from a straightforward verification using (9) on (11) and (12). $\qquad\qquad \square$

The Factorization Manifold Embedding Theorem implies the geometry of polynomial factorization: *The subset $\mathscr{R}_{\mathbf{m}}$ of degree $\mathbf{m}$ polynomials is a disjoint union*
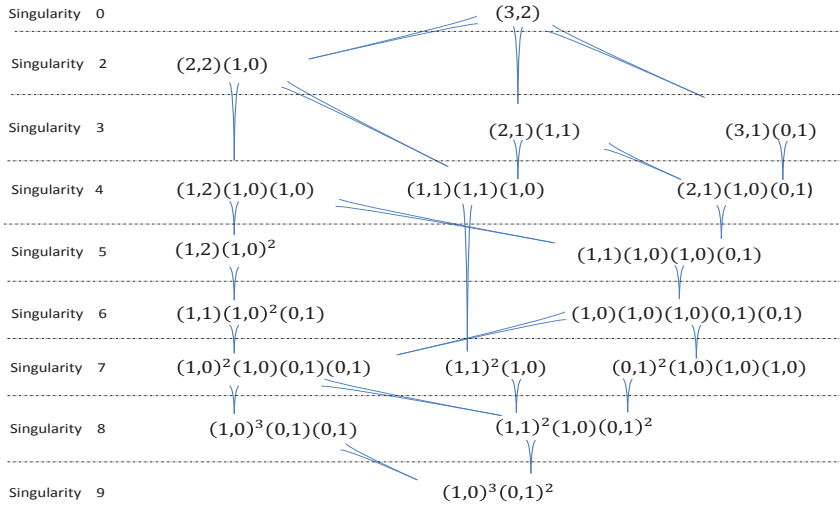
Fig. 1 Stratification of factorization manifolds in $\mathscr{R}_{(3,2)}$, where $\mathfrak{N} \prec \mathfrak{M}$ indicates $\mathscr{F}_{\mathfrak{N}} \subset \overline{\mathscr{F}_{\mathfrak{M}}}$.

of factorization manifolds that are topologically stratified in such a way that every factorization manifold of positive singularity is embedded in the closure of a factorization manifold of lower singularity. As an example, Figure 1 illustrates such a stratification among all the factorization manifolds through corresponding factorization structures in $\mathscr{R}_{(3,2)}$.

We define the distance between a polynomial and a factorization manifold

$$dist\left(f, \mathscr{F}_{\mathfrak{M}}\right) \quad = \quad \inf_{g \in \mathscr{F}_{\mathfrak{M}}} \left(||f - g||\right). \tag{13}$$

Let $\mathfrak{N}$ be the factorization structure of f. The distance $dist\left(f, \mathscr{F}_{\mathfrak{M}}\right) = 0$ if and only if $f \in \overline{\mathscr{F}_{\mathfrak{M}}}$, which is equivalent to $\mathfrak{N} \prec \mathfrak{M}$ by the Factorization Manifold Embedding Theorem. As a consequence, the native manifold $\mathscr{F}_{\mathfrak{N}}$ of f distinguishes itself as the unique factorization manifold that is of the *highest singularity* (i.e. highest codimension) among all the factorization manifolds having a distance zero to f. More precisely, a polynomial f belongs to a factorization manifold $\mathscr{F}_{\mathfrak{N}}$ if and only if, in $\mathcal{R}_{\mathbf{m}}$,

$$codim(\mathscr{F}_{\mathfrak{N}}) \quad = \\ \max \left\{ codim(\mathscr{F}_{\mathfrak{M}}) \;\middle|\; deg(\mathfrak{M}) = deg(f) = \mathbf{m} \text{ and } dist\left(f, \mathscr{F}_{\mathfrak{M}}\right) = 0 \right\}.$$

On the other hand, a polynomial $\tilde{f} \in \mathscr{F}_{\mathfrak{N}}$ with $\mathfrak{N} \nprec \mathfrak{M}$ implies that the distance $dist\left(\tilde{f}, \mathscr{F}_{\mathfrak{M}}\right)$ is positive. Since there are finitely many factorization manifolds, there exists a minimum positive distance

$$\theta_{\tilde{f}} \quad = \quad \min_{\mathscr{F}_{\mathfrak{M}} \not\ni \tilde{f}} dist\left(\tilde{f}, \mathscr{F}_{\mathfrak{M}}\right) \quad > \quad 0. \tag{14}$$

The constant $\theta_{\tilde{f}}$ is the *critical gap* of $\tilde{f}$ from unembedded singularities and it is the very window of opportunity for numerical factorization. When the polynomial $\tilde{f} \in \mathscr{F}_{\mathfrak{N}}$ is represented by an empirical version f with a small perturbation $\|f - \tilde{f}\| < \frac{1}{2}\theta_{\tilde{f}}$, the underlying factorization structure can still be identified by the following lemma.

Lemma 4 *Let* $\tilde{f}$ *be a polynomial with a factorization structure* $\mathfrak{N}$ *with* $\theta_{\tilde{f}}$ *be given in* (14)*. For any empirical data* f *of* $\tilde{f}$ *satisfying* $\|f - \tilde{f}\| < \frac{1}{2}\theta_{\tilde{f}}$*, the factorization structure* $\mathfrak{N}$ *of* $\tilde{f}$ *is uniquely identifiable using the data* f *by*

$$
\begin{aligned}
codim(\mathscr{F}_{\mathfrak{N}}) \;\; &= \\
\max\big\{ codim(\mathscr{F}_{\mathfrak{M}}) \mid \;\; &deg(\mathfrak{M}) \;=\; deg(f) = \mathbf{m} \;\; and \;\; dist\big(f, \mathscr{F}_{\mathfrak{M}}\big) < \epsilon \big\} \quad (15)
\end{aligned}
$$

in $\mathcal{R}_{\mathbf{m}}$ *for any* ϵ *satisfying* $dist\big(f, \mathscr{F}_{\mathfrak{N}}\big) < \epsilon < \frac{1}{2}\theta_{\tilde{f}}$*.*

PROOF. A straightforward verification. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

In summary, singular polynomials form factorization manifolds with positive codimensions and nonsingular polynomials form an open dense subset in $\mathcal{R}_{\mathbf{m}}$. Those factorization manifolds topologically stratify in such a way that every singular manifold belongs to the closures of some manifolds of lower singularities. Almost all tiny perturbations on a singular polynomial alter its factorization structure in such a way that the singularity reduces and never increases. There is a gap from any singular polynomial to higher singularity and this gap ensures the lost factorization structure can be recovered by finding the highest singularity manifold nearby if the perturbation is small. As a result, identifying the factorization structure is well-posed as an optimization problem.

5 The notion of numerical factorization

We shall rigorously formulate the concept of the numerical factorization to remove the ill-posedness of the conventional factorization, and to achieve the main objective of recovering the exact factorization accurately using the imperfect empirical data. The numerical factorization should approximate the underlying factorization with an accuracy the data deserve. The following problem statement gives a precise description of the problem that numerical factorization is intended to solve.

PROBLEM 51 (Numerical Factorization Problem) *Let* f *be a polynomial as the empirical data of an underlying polynomial* $\tilde{f}$ *whose irreducible factorization* $\tilde{\alpha}\tilde{f}_1 \cdots \tilde{f}_r$ *is to be computed. Assuming the data error* $\|f - \tilde{f}\|$ *is sufficiently small, find an irreducible factorization* $\alpha f_1 \cdots f_k$ *of a certain polynomial* $\hat{f}$ *such that both the backward error and forward error are in the order of data error and the unit round-off:*

$$
\begin{aligned}
\big\| f - \alpha f_1 \cdots f_k \big\| \;\; &= \;\; O\big(\|f - \tilde{f}\| + \mathfrak{u}\big) & (16) \\
dist\big(\alpha f_1 \cdots f_k, \;\; \tilde{\alpha}\tilde{f}_1 \cdots \tilde{f}_r\big) \;\; &= \;\; O\big(\|f - \tilde{f}\| + \mathfrak{u}\big). & (17)
\end{aligned}
$$

where $\mathfrak{u}$ *is the unit round-off in the floating point arithmetic.*

Problem 51 goes a step further from the Open Problem 1 in [16] in which only the backward error is required to be small. Notice that the factorization $\alpha f_1 \cdots f_k$ in (17) does not need to have the same number of factors as the underlying factorization $\tilde{\alpha} \tilde{f}_1 \cdots \tilde{f}_r$. When $r \neq k$ and the distance is small in (17), some of the factors are approximately constants.

The assumption of the data error $\|f - \tilde{f}\|$ to be sufficiently small should not be confused with the local convergence of an iterative method. It is not a requirement of having a good approximation of the *solution* as local convergence implies. Rather, it assumes we are not trying to solve a *problem* that is too faraway to be relevant. From conventional factorization where data error is required to be zero, allowing sufficiently small data perturbation while still achieving a high accuracy is the significant advancement that the numerical factorization is intended to achieve. Basic data integrity is required in all scientific studies, albeit almost always implicitly. It needs to be explicitly stated here because we are in a frontier of solving discontinuous problems for which the difference is profound between exact data and perturbed ones, and the latter may be all we have.
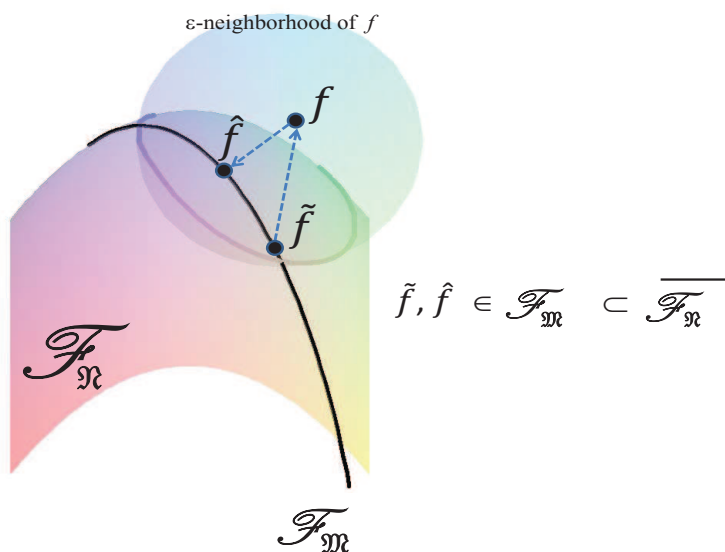
Fig. 2 Illustration of the numerical factorization of f

Let $\tilde{f}$ be the polynomial in Problem 51 with the factorization structure $\mathfrak{M}$ and f be its empirical data representation, as illustrated in Figure 2. By the Factorization Manifold Embedding Theorem, the data polynomial f is away from the native factorization manifold $\mathscr{F}_{\mathfrak{M}}$ with a *reduced singularity*. Note that the data polynomial f is also near all the factorization manifolds $\mathscr{F}_{\mathfrak{N}}$ with $\mathfrak{M} \prec \mathfrak{N}$ and the native manifold $\mathscr{F}_{\mathfrak{M}}$ is not the nearest in distance but highest in singularity by Lemma 4. Upon identifying the factorization structure $\mathfrak{M}$, it

is then natural to calculate the exact irreducible factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$ of the polynomial $\hat{f} \in \mathscr{F}_{\mathfrak{M}}$ that is the nearest to f and designate it as the numerical irreducible factorization of f since Corollary 5 suggests that the (exact) irreducible factorization of $\hat{f}$ approximates that of $\tilde{f}$. The following definition is the detailed formulation.

Definition 7 (Numerical Factorization) *For a given polynomial* f *and a backward error tolerance* $\epsilon > 0$, *we say* $\alpha f_1 f_2 \cdots f_s$ *is a* **numerical irreducible factorization of** f **within** ϵ *if* $\alpha f_1 f_2 \cdots f_s \in \mathscr{F}_{\mathfrak{M}}$ *is an irreducible factorization and*

$$\min_{\gamma \in \mathbb{C}} \left\| f - \gamma f_1 f_2 \cdots f_s \right\| = \min_{g \in \mathscr{F}_{\mathfrak{M}}} \| f - g \| = \mathit{dist}\big(f, \mathscr{F}_{\mathfrak{M}}\big) < \epsilon, \qquad (18)$$

where $\mathfrak{M}$ *is the factorization structure of the degree* $\mathbf{m} = \mathit{deg}(f)$ *with the highest singularity*

$$\mathit{codim}(\mathscr{F}_{\mathfrak{M}}) = \max \big\{ \mathit{codim}(\mathscr{F}_{\mathfrak{N}}) \,\big|\, \mathit{deg}(\mathfrak{N}) = \mathbf{m} \ \text{and} \ \mathit{dist}(f, \mathscr{F}_{\mathfrak{N}}) < \epsilon \big\} \qquad (19)$$

in $\mathcal{R}_{\mathbf{m}}$. *We call* $\alpha f_1^{k_1} \cdots f_r^{k_r}$ *a* **numerical irreducible squarefree factorization of** f **within** ϵ *if it is squarefree and it is a numerical irreducible factorization of* f *within* ϵ.

We shall use the abbreviated term **numerical factorization** for either the numerical irreducible factorization or the numerical irreducible squarefree factorization when the distinction is insignificant in the context. The formulation of the numerical factorization follows the same "three-strikes" principles that have been effectively applied to the regularization of other ill-posed algebraic problems [31]: The numerical factorization of f is the exact factorization of a nearby polynomial $\hat{f}$ within a backward error tolerance ϵ (backward nearness principle). The nearby polynomial $\hat{f}$ is of the highest singularity among all the polynomials in the ϵ-neighborhood of f (maximum singularity principle). The nearby polynomial $\hat{f}$ is the nearest polynomial to the given f among all the polynomials with the same singularity as $\hat{f}$ (minimum distance principle).

The error tolerance ϵ in Definition 7 depends on the particular application, the hardware precision, the underlying polynomial $\tilde{f}$ and the data error $\|f - \tilde{f}\|$. The interval for setting ϵ will be established in the Numerical Factorization Theorem in §6. Notice that a polynomial f can easily have different numerical factorizations within different error tolerances approximating different factorizations (c.f. Example 5 in §9).

6 Regularity and sensitivity of numerical factorization

As a concept attributed to Jacques S. Hadamard, a mathematical problem is well-posed if its solution holds existence, uniqueness and continuity with respect to data. Furthermore, Lipschitz continuity of the solution is crucial for numerical computation as it implies a finite sensitivity with respect to data perturbations and round-off. With the geometry established in §3 and §4 , the well-posedness of numerical factorization is a direct consequence of the Tubular Neighborhood Theorem, which is one of the fundamental results in differential topology. The following elementary version of the Tubular Neighborhood Theorem is adapted from its abstract form for complex analytic manifolds in $\mathbb{C}^n$.

Lemma 5 (Tubular Neighborhood Theorem) [33] *Every complex analytic manifold is contained in a tubular neighborhood. More precisely, for every complex analytic manifold Π in $\mathbb{C}^n$, there is an open subset Ω of $\mathbb{C}^n$ containing Π and a projection mapping $\pi : \Omega \longrightarrow \Pi$ such that, for every $\mathbf{z} \in \Omega$, its projection $\pi(\mathbf{z}) \in \Pi$ is the unique distance-minimization point from $\mathbf{z}$ to Π, namely $\|\pi(\mathbf{z}) - \mathbf{z}\|_2 = \min_{\mathbf{u} \in \Pi} \|\mathbf{u} - \mathbf{z}\|_2$. Furthermore, the mapping π is locally Lipschitz continuous.*

We can now establish the main theorem, which asserts the properties that are desirable from the numerical factorization as formulated in Definition 7, provides a complete regularization and, in essence, achieves the objectives of numerical factorization in Problem 51.

Theorem 8 (Numerical Factorization Theorem) *Let $\tilde{f} \in \mathscr{R}_{\mathbf{m}}$ be a polynomial with its critical gap $\theta_{\tilde{f}}$ as in (14) and an irreducible factorization $\tilde{\alpha} \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_k$. Then $\theta_{\tilde{f}} > 0$ and the following properties of numerical factorization hold.*

(i)   Conventional factorization is a special case of numerical factorization: *The numerical factorization of $\tilde{f}$ within any $\epsilon \in (0, \theta_{\tilde{f}})$ is identical to the exact irreducible factorization of $\tilde{f}$.*

(ii) Computing numerical factorization is a well-posed problem: *There is a neighborhood $\Omega_{\tilde{f}}$ of $\tilde{f}$ in $\mathcal{R}_{\mathbf{m}}$ such that every $f \in \Omega_{\tilde{f}}$ is associated with a constant $\delta_f \le \|f - \tilde{f}\|$ such that the numerical factorization $\alpha f_1 f_2 \cdots f_l$ of f uniquely exists within ϵ for all $\epsilon \in \left(\delta_f, \frac{1}{2}\theta_{\tilde{f}}\right)$ and is Lipschitz continuous with respect to f.*

(iii) Numerical factorization is backward accurate: *For every $f \in \Omega_{\tilde{f}}$ and ϵ in $\left(\delta_f, \frac{1}{2}\theta_{\tilde{f}}\right)$, the numerical factorization $\alpha f_1 f_2 \cdots f_l$ of f within ϵ satisfies*

$$\left\| f - \alpha f_1 f_2 \cdots f_l \right\| \le \|f - \tilde{f}\|. \tag{20}$$

(iv) The conventional factorization can be accurately recovered from empirical data: *For every $f \in \Omega_{\tilde{f}}$ as empirical data of $\tilde{f}$ and $\epsilon \in \left(\delta_f, \frac{1}{2}\theta_{\tilde{f}}\right)$, the numerical factorization $\alpha f_1 f_2 \cdots f_l$ of f within ϵ has the identical structure as $\tilde{\alpha} \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_k$ and*

$$\mathit{dist}\left(\alpha f_1 f_2 \cdots f_l,\ \tilde{\alpha} \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_k\right) < \eta_{\tilde{f}} \|f - \tilde{f}\|. \tag{21}$$

where $\eta_{\tilde{f}} > 0$ is a constant depends on f.

PROOF. Let $\mathfrak{M}$ denote the factorization structure of $\tilde{f}$. Then $\mathscr{F}_{\mathfrak{M}}$ is the manifold of the highest singularity within $\epsilon \in (0, \theta_{\tilde{f}})$ of $\tilde{f}$, and $\tilde{f}$ itself is the polynomial of minimum distance zero on $\mathscr{F}_{\mathfrak{M}}$ from $\tilde{f}$, and thus (i) holds.

Let Σ be the tubular neighborhood of $\mathscr{F}_{\mathfrak{M}}$ described in Lemma 5 and let $\Omega_{\tilde{f}} \subset \Sigma$ be a neighborhood of $\tilde{f}$ such that every $f \in \Omega_{\tilde{f}}$ satisfies $\|f - \tilde{f}\| < \frac{1}{2}\theta_{\tilde{f}}$. Set $\delta_f = \mathit{dist}\left(f, \mathscr{F}_{\mathfrak{M}}\right)$. Then, for every $\epsilon \in (\delta_f, \frac{1}{2}\theta_{\tilde{f}})$ the equality (19) holds since $\mathit{dist}\left(f, \mathscr{F}_{\mathfrak{N}}\right) > \frac{1}{2}\theta_{\tilde{f}} > \epsilon$ for every $\mathfrak{N} \not\succ \mathfrak{M}$. By the Tubular Neighborhood Theorem, there exists a unique $\hat{f} = \pi(f) \in \mathscr{F}_{\mathfrak{M}}$ with minimal distance to f. As a result, the

numerical factorization of f uniquely exists as the exact irreducible factorization of $\hat{f}$, and the numerical factorization is locally Lipschitz continuous since π is locally Lipschitz continuous along with Corollary 5, leading to part (ii).

Part (iii) is true since $\|f - \hat{f}\| \leq \|f - \tilde{f}\|$. The Lipschitz continuity of the numerical factorization also implies (21) and part (iv).   □

In simpler terms, Numerical Factorization Theorem ensures that every factorable polynomial $\tilde{f}$ is allowed to be perturbed while its factorization can still be recovered as long as the empirical data f is still in the neighborhood $\Omega_{\tilde{f}}$. For each data representation f of $\tilde{f}$, there is a window $(\delta_f, \frac{1}{2}\theta_{\tilde{f}})$ for setting the error tolerance ϵ for recovering the factorization of $\tilde{f}$. The fact that the lower bound δ_f of the error tolerance ϵ is no larger than the data error $\|f - \tilde{f}\|$ is significant in practical computation: If a data error bound $\eta > 0$ for $\|f - \tilde{f}\|$ is known or can be estimated in an application, the error tolerance can be set at $\epsilon = \eta$ or a moderate multiple of the unit round-off, whichever is larger. The upper bound $\frac{1}{2}\theta_{\tilde{f}}$ appears to be difficult to estimate but not needed as long as it is not too small.

With a proper error tolerance ϵ, the numerical factorization of the data f within ϵ approximates the exact factorization of the underlying polynomial $\tilde{f}$ with an accuracy in the same order of the data accuracy. Namely, the numerical factorization we formulated in Definition 7 achieves the objective of the numerical factorization problem as specified in Problem 51. Furthermore, computing the numerical factorization is a well-posed problem with a finite sensitivity that can be established in the following theorem.

The assumption $\tilde{f} \in \mathscr{R}_{\mathbf{m}}$ in Theorem 8 implies that the degree of the data polynomial f is the same as the degree of the underlying polynomial $\tilde{f}$. If the data error is sufficiently small, the degree of the data polynomial is at least equal to the degree of the underlying polynomial. When the perturbation is of a higher degree but sufficiently small in magnitude in monomial basis, the coefficients of those extra monomials are below the error tolerance. As illustrated in Example 2 in §9, higher degree perturbations in monomial basis generally are negligible in practical computation.

The condition $f \in \Omega_{\tilde{f}}$ in Theorem 8 is to ensure the data error to be "sufficiently small" as one of main assumptions of this paper. As Tubular Neighborhood Theorem dictates, the given polynomial needs to be close to manifold since the projection to the manifold may not be unique otherwise. As a result, accurate recovery of the factorization cannot be guaranteed from perturbed data with large errors. Theories and algorithms for numerical factorizations with data perturbations of large magnitudes, higher degrees and in polynomial bases other than monomials are open problems for further studies.

Theorem 9 (Numerical Factorization Sensitivity Theorem) *Let* $\alpha f_1^{k_1} \cdots f_r^{k_r}$ *be the numerical factorization of* f *within certain* ϵ *and* g *be sufficiently close to* f *so that its numerical factorization within* ϵ *can be written as and* $\gamma g_1^{k_1} \cdots g_r^{k_r}$ *with* $deg(g_j) = deg(f_j) = \mathbf{m}_j$ *for* $j = 1, 2, \ldots, r$. *Further assume* $\mathcal{J}(\cdot)$ *is as defined in* (7) *where* $\mathbf{b}_j \in \mathbb{C}^{\langle \mathbf{m}_j \rangle}$ *with* $\|\mathbf{b}_j\|_2 = 1$ *and* $\mathbf{b}_j \cdot [\![f_j]\!] = 1$ *for* $j = 1, \ldots, r$. *Then*

$$\limsup_{g \to f} \frac{dist\left(\alpha f_1^{k_1} \cdots f_r^{k_r}, \ \gamma g_1^{k_1} \cdots g_r^{k_r}\right)}{\|f - g\|} \ \leq \ \eta \left\| \mathcal{J}(\alpha, [\![f_1]\!], \ldots, [\![f_r]\!])^+ \right\|_2 \ < \ \infty \quad (22)$$

where η is a constant associated with f and $\mathbf{b}_1, \ldots, \mathbf{b}_r$.

PROOF. A straightforward verification using Theorem 2 in [33] and Lemma 1. $\square$

The inequality (22) depends on the choices of the specific representative factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$ in the equivalent class of factorizations and the scaling vectors $\mathbf{b}_1, \ldots, \mathbf{b}_r$. Independent of those choices, we define the positive real number

$$\kappa_\epsilon(f) \quad := \quad \inf \Big\{ \big\| \mathcal{J}(\beta, [\![h_1]\!], \ldots, [\![h_r]\!])^+ \big\|_2 \,\Big|\, \beta h_1^{k_1} \cdots h_r^{k_r} \sim \alpha f_1^{k_1} \cdots f_r^{k_r},$$
$$\mathbf{b}_j \in \mathbb{C}^{\langle \mathbf{m}_j \rangle}, \ \|\mathbf{b}_j\|_2 = 1, \ \mathbf{b}_j \cdot [\![h_j]\!] = 1, \ j = 1, \ldots, r \Big\} \qquad (23)$$

as the **condition number of the numerical factorization** of f within ϵ where $\alpha f_1^{k_1} \cdots f_r^{k_r}$ is a numerical squarefree irreducible factorization of f within ϵ. From Lemma 2, this condition number is finite since the factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$ of f is squarefree, and $\kappa_\epsilon(f)$ becomes large when the Jacobian $\mathcal{J}(\beta, [\![h_1]\!], \ldots, [\![h_r]\!])^+$ is near rank-deficient when two of the factors $h_1, \ldots, h_r$ are a small perturbation away from having nonconstant GCD. Consequently, the nature of the computation stability of numerical factorization become apparent: *The numerical factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$ of f is ill-conditioned if there exist two factors f_i and f_j that are near non-coprime polynomials so that a small perturbation of f can increase the singularity above that of $\hat{f} = \alpha f_1^{k_1} \cdots f_r^{k_r}$.*

7 On the numerical squarefree factorization

Every polynomial f has a unique squarefree factorization $\alpha f_1^{k_1} \cdots f_r^{k_r}$ where $f_1, \ldots, f_k$ are pairwise coprime squarefree polynomials. Such squarefree factorizations are important in its own right and usually easier to compute than irreducible factorizations. Our numerical factorization test algorithm and implementation start with finding a numerical squarefree factorization followed by numerical irreducible factorizations of the squarefree components. Naturally, the notion of numerical squarefree factorization and its properties are in question.

Similar to (irreducible) factorization structure, we can define a *squarefree factorization structure* $\mathfrak{N} = \mathbf{n}_1^{k_1} \cdots \mathbf{n}_r^{k_r}$ of polynomials having a squarefree factorization $f = \alpha f_1^{k_1} \cdots f_r^{k_r}$ where $k_1 \leq \cdots \leq k_r$ where $f_j \in \mathscr{R}_{\mathbf{n}_j}$ is squarefree for $j = 1, \ldots, r$ and pairwise coprime. Also let $\mathscr{S}_{\mathfrak{N}}$ denote the collection of polynomials in $\mathscr{R}_{\mathbf{m}}$ having a squarefree factorization structure $\mathfrak{N}$. Notice that Lemma 2 applies to squarefree factorizations since the irreducibility of factors is not required. It is also a straightforward verification that Lemma 3 and Corollary 1 still hold for $\mathscr{S}_{\mathfrak{N}}$. As a result, the subset $\mathscr{S}_{\mathfrak{N}}$ is also a complex analytic manifold in $\mathcal{R}_{\mathbf{d}}$ of codimension

$$codim(\mathscr{S}_{\mathfrak{N}}) \quad = \quad \langle \mathbf{d} \rangle - \Big(\langle \mathbf{n}_1 \rangle + \cdots + \langle \mathbf{n}_r \rangle + 1 - r \Big).$$

The embedding properties of squarefree factorization manifolds hold as well.

Similar to Definition 7, we can formulate the *numerical squarefree factorization* of a polynomial f within an error tolerance ϵ as the exact factorization of a polynomial $\hat{f} \in \mathscr{S}_{\mathfrak{N}}$ where $\mathscr{S}_{\mathfrak{N}}$ is the squarefree factorization manifold of the highest codimension among all manifolds intersecting the ϵ-neighborhood of f

and $\hat{f}$ is the nearest polynomial from f on $\mathscr{S}_{\mathfrak{N}}$. Such a numerical squarefree factorization is a generalization of the conventional exact squarefree factorization and accurate approximation to the exact squarefree factorization of the underlying polynomial $\tilde{f}$. Furthermore its computation is a well-posed problem with a finite sensitivity measure.

For a unit vector $\mathbf{z} = (z_1, \ldots, z_\ell) \in \mathbb{C}^\ell$, let $\partial_\mathbf{z} p$ denote the directional derivative of $p \in \mathbb{C}[x_1, \ldots, x_\ell]$ along (the direction of) $\mathbf{z}$, namely

$$\partial_\mathbf{z} p \;\; = \;\; z_1 \frac{\partial p}{\partial x_1} + \cdots + z_\ell \frac{\partial p}{\partial x_\ell}.$$

The following lemma is the basis for the numerical squarefree factorization.

Lemma 6 *Every $f \in \mathbb{C}[x_1, \ldots, x_\ell]$ has a squarefree factorization $\alpha h_1^{k_1} \cdots h_r^{k_r}$ with non-constant factors $h_1, \ldots, h_r$ and distinct multiplicities $k_1, \ldots, k_r \geq 1$. Furthermore, for almost all unit vectors $\mathbf{z} \in \mathbb{C}^\ell$,*

$$gcd\,(v,\, l \cdot \partial_\mathbf{z} v - w) \;\; = \;\; \begin{cases} h_j & for \;\; l = k_j \;\; with \;\; j = 1, \ldots, r \\ 1 & if \;\; l \notin \{k_1, \ldots, k_r\} \end{cases} \tag{24}$$

where v and w are cofactors of $u = gcd\,(f, \partial_\mathbf{z} f)$ such that $f = uv$ and $\partial_\mathbf{z} f = uw$.

PROOF. The existence of $h_1, \ldots, h_r$ is obvious. For almost all $\mathbf{z} \in \mathbb{C}^\ell$, $\partial_\mathbf{z} h_j \neq 0$ for $j = 1, \ldots, r$. Thus $v \sim h_1 \cdots h_r$ and $w \sim \sum_{j=1}^{r} k_j (\partial_\mathbf{z} h_j) \prod_{i \neq j} h_i$, leading to (24). $\qquad\square$

The numerical squarefree factorization can be computed by a sequence of numerical greatest common divisor replacing the exact GCD in (24).

By Lemma 2, the Jacobian of the mapping ϕ in (6) is injective at the least squares solution of $\phi(\beta, [\![g_1]\!], \ldots, [\![g_r]\!]) = ([\![f]\!], 1, \ldots, 1)$, implying the Gauss-Newton iteration locally converges to this least squares solution if the data f and the initial iterate are sufficiently accurate.

Due to its similarity with the numerical irreducible squarefree factorization, we omit the detailed elaboration of the numerical squarefree factorization in this paper.

8 Computation of numerical factorizations

Overall, computing the numerical factorization consists of two stages. The first stage identifies the factorization structure along with initial approximations of the numerical factors. In the second stage, the numerical factors are refined to minimize the distance from the given polynomial to the manifold associated with the factorization structure.

In the first stage, the factorization structure can be computed by a sequence numerical squarefree factorizations,rank-revealing of the Ruppert matrices [9, 18, 23], generalized eigenvalue computation and numerical greatest common divisor calculation. Initial approximations of the numerical factors are obtained as by-products. The computation requires standard numerical linear algebra operations with complexities of $O(n^3)$ where n is the number of relevant monomials of the polynomial and the numerical factors.

In the second stage, the initial factor approximations $p_1, \ldots, p_r$ of degrees $\mathbf{m}_1, \ldots, \mathbf{m}_r$ can be scaled to unit norms $\|p_1\| = \cdots = \|p_r\| = 1$. The mapping ϕ in (6) becomes well defined by setting up the scaling vectors $\mathbf{b}_i = [\![p_i]\!]$ for $i = 1, \ldots, r$, The second stage of the numerical factorization algorithm is essentially the process of solving for the least squares solution to the overdetermined nonlinear system

$$\phi(\mathbf{z}) \;=\; ([\![f]\!], 1, \ldots, 1), \quad \text{with} \quad \mathbf{z} \in \mathbb{C} \times \mathbb{C}^{\langle \mathbf{m}_1 \rangle} \times \cdots \times \mathbb{C}^{\langle \mathbf{m}_r \rangle} \qquad (25)$$

using the Gauss-Newton iteration

$$\mathbf{z}_{j+1} \;=\; \mathbf{z}_j - \mathcal{J}(\mathbf{z}_j)^+ \, \phi(\mathbf{z}_j), \quad j = 0, 1, \ldots. \qquad (26)$$

where $\mathcal{J}(\mathbf{z})^+$ is the pseudo-inverse of the Jacobian $\mathcal{J}(\mathbf{z})$ of $\phi(\mathbf{z})$ given in (7) and $\mathbf{z}_0 = (\alpha, [\![p_1]\!], \ldots, [\![p_r]\!])$.

Detailed discussion on the Gauss-Newton iteration and conditions for its convergence can be found in [2, 26, 31, 33]. In a nutshell, the iteration (26) locally converges to the least squares solution $\mathbf{z}_*$ that is the point satisfying

$$\left\| \phi(\mathbf{z}_*) - ([\![f]\!], 1, \ldots, 1) \right\|_2 \;=\; \min_{\mathbf{z} \in \mathbb{C} \times \mathbb{C}^{\langle \mathbf{m}_1 \rangle} \times \cdots \times \mathbb{C}^{\langle \mathbf{m}_r \rangle}} \left\| \phi(\mathbf{z}) - ([\![f]\!], 1, \ldots, 1) \right\|_2 \quad (27)$$

if both the residual $\left\| \phi(\mathbf{z}_*) - ([\![f]\!], 1, \ldots, 1) \right\|_2$ and the initial error $\|\mathbf{z}_0 - \mathbf{z}_*\|_2$ are small.

Lemma 7 *Let* $\alpha f_1^{k_1} \cdots f_r^{k_r}$ *be the numerical squarefree irreducible factorization of* f *within* ϵ *with* $\mathbf{m}_j = deg(f_j)$ *for* $j = 1, \ldots, r$. *Then, for almost all unit vectors* $\mathbf{b}_i \in \mathbb{C}^{\langle \mathbf{m}_i \rangle}$, $i = 1, \ldots, r$, *there is a factorization* $\alpha_* f_{*1}^{k_1} \cdots f_{*r}^{k_r} \sim \alpha f_1^{k_1} \cdots f_r^{k_r}$ *such that* $\mathbf{z}_* = \left(\alpha_*, [\![f_{*1}]\!], \ldots, [\![f_{*r}]\!] \right)$ *is the least squares solution to the equation* (25) *with residual*

$$\left\| \phi(\mathbf{z}_*) - ([\![f]\!], 1, \ldots, 1) \right\|_2 \;=\; \| f - \alpha f_1^{k_1} \cdots f_r^{k_r} \| \qquad (28)$$

where ϕ *is defined as in* (6).

PROOF. Clearly $\left\| \phi(\mathbf{z}) - ([\![f]\!], 1, \ldots, 1) \right\|_2 \geq \| f - \alpha f_1^{k_1} \cdots f_r^{k_r} \|$ by Definition 7. On the other hand, setting $f_{*i} = \frac{1}{\mathbf{b}_i \cdot [\![f_i]\!]} f_i$ for $i = 1, \cdots, r$ and an appropriate α_* yields (28). $\qquad \square$

Under the main condition that the Jacobian $\mathcal{J}(\mathbf{z}_*)$ is of full rank, the Gauss-Newton iteration converges locally [31]. The local convergence of the Gauss-Newton iteration requires two conditions: The initial iterate $\mathbf{z}_0$ must be near the least squares solution $\mathbf{z}_*$ and the residual $\|\phi(\mathbf{z}_*) - ([\![f]\!], 1, \ldots, 1)\|_2$ must be sufficiently small. From (28), the residual $\|\phi(\mathbf{z}_*) - ([\![f]\!], 1, \ldots, 1)\|_2$ is bounded by the data error $\| f - \tilde{f} \|$. As a result, the residual requirement will be satisfied if the data error is sufficiently small.

The convergence of the two-staged method relies on the basic assumption of the problem data being sufficiently close to the underlying polynomial. Accurate data lead to sufficiently small residual and an initial iterate being close to the least squares solution so that the local convergence conditions of Gauss-Newton iteration are satisfied based on the alpha-theory [2, 26]. Conversely, numerical factorization from poor data is not guaranteed to succeed partly because the Gauss-Newton iteration is locally convergent. The algorithmic and technical details of the numerical factorization are beyond the scope of this paper and will be elaborated in a separate report.

9 Implementation, software and sample results

Our numerical factorization test algorithm is implemented for both univariate and multivariate polynomials as a function `PolynomialFactor` in the Matlab package `NAClab` for numerical algebraic computation as an upgrade and an expansion from its predecessor `Apalab` [32]. The entire `NAClab` package is freely available[1], including numerical factorization, numerical rank-revealing, numerical computation of multiplicity structure at zeros of nonlinear systems, numerical greatest common divisors, etc. We shall present several sample results highlighting the theories in this paper and the major improvement areas of our test algorithm and the resulting software: Efficiency, accuracy, versatility and user friendliness. The numerical factorization software is made available for other researchers to experiment with. All the tests are carried out on a Samsung Series 7 XE700T1A tablet computer with 4GB memory and Intel i5-2467M CPU at 1.60 GHz running on Windows 7 64-bit operating system. The test log and relevant Matlab/Maple scripts can be downloaded online[2].

The Matlab package `NAClab` provides a user friendly interface for numerical algebraic computations. Polynomials can be entered and output as intuitive strings for casual users. The function `PolynomialFactor` can be conveniently executed by setting an error tolerance at or slightly above the magnitude of the data error.

Example 2 (Basic software application) Consider the polynomial

$$
\begin{aligned}
f = {} & -30 + 20\,x + 18\,x^2 - 12\,x^3 + 12.000007\,x\,y - 8\,x^2\,y + 3\times10^{-7}\,x^3\,y - 5\,y^2 \\
& + 3\,x^2\,y^2 + 2\,x\,y^3 \\
\equiv {} & (3\,x^2 + 2\,x\,y - 5)\,(y^2 - 4\,x + 6) + 0.0000003\,x^3\,y + 0.000007\,x\,y
\end{aligned}
$$

as the data polynomial of $(3\,x^2 + 2\,x\,y - 5)\,(y^2 - 4\,x + 6)$ with a perturbation of magnitude about 7×10^{-6}. Setting the error tolerance slightly larger as, say 10^{-5}, a call of the functionality POLYNOMIALFACTOR in NACLAB yields

```
>> p = '-30+20*x+18*x^2-12*x^3+12.000007*x*y-8*x^2*y+3e-07*x^3*y-5*y^2+3*x^2*y^2+2*x*y^3'
>> PolynomialFactor(p,1e-5,'row')
ans =
(-30.0000005908641) * (1-0.6*x^2-0.400000158490569*x*y) * (1-0.666666625730982*x+0.166666656432746*y^2)
```

The resulting numerical factorization has the same order of accuracy as the data. Notice that the degree of the perturbation is not limited to the degree of the underlying polynomial.

Example 3 (Univariate factorization) Accurate factorization of univariate polynomials with multiple roots has been a challenge in numerical computation. Conventional software functions for polynomial root-finding, such as Matlab `roots` and Maple `fsolve` can not factor such polynomial accurately and output scattered root clusters. For example, let

$$
\begin{aligned}
f(x) \;=\; & \\
=\; & x^{100} - {}_{222.222222222222}\,x^{99} + \cdots - {}_{8.53544016536406\ 10^{30}}\,x + {}_{1.47799829703274\ 10^{29}} \quad (29) \\
\approx\; & \bigl(x - {}_{4.444444444444444}\bigr)^{10}\bigl(x - {}_{3.333333333333333}\bigr)^{20}\bigl(x - {}_{2.222222222222222}\bigr)^{30} \times \\
& \bigl(x - {}_{1.111111111111111}\bigr)^{40}
\end{aligned}
$$

---

[1]   `http://homepages.neiu.edu/∼naclab.html`

[2]   `http://homepages.neiu.edu/∼zzeng/NumFactorTests.zip`

In contrast, our `PolynomialFactor` is an advanced polynomial root-finder that is capable of accurate computation for multiple roots without extending machine precision even if the coefficients are perturbed. On this example, our `PolynomialFactor` yields a factorization containing accurate roots and multiplicities:

```
>> PolynomialFactor(f,1e-10,'row')
ans =
(x-4.44444444445)^10 * (x-3.33333333333)^20 * (x-2.22222222222)^30 * (x-1.11111111111)^40
```

This is a substantial improvement over its predecessor [30].   □

Since available software implementations for multivariate factorizations are built on different platforms, based on different notions of numerical factorizations, and with different designing emphases, comprehensive comparisons are not feasible. Among them, Maple `factor` is built for the exact factorization. Developed by Vershelde, the package `PHC`[29] is a general-purpose polynomial system solver whose factorization option `-f` is perhaps the first implemented numerical factorization software. This `PHC` option initiates the implementation of a factorization algorithm [27] in numerical computation based on the homotopy continuation method. The Maple code `appfac` is developed by Kaltofen, May, Yang and Zhi [19] and the algorithm uses similar reducibility test based on [9, 23], which appears to be superior in factoring polynomial with highly perturbed data. The computing examples in the remainder of this section are designed to showcase the differences and improvement areas of our test algorithm and implementation.

Example 4 (Stewart-Gough Platforms) In [27], the authors tested three polynomials derived from the Stewart-Gough platform manipulator in mechanical engineering:

$$\begin{cases} g_1 = F_1(q_0, q_1, q_2, q_3)\,(q_0^2 + q_1^2 + q_2^2 + q_3^2)^3 \\ g_2 = F_2(q_0, q_1, q_2, q_3)\,(q_0^2 + q_1^2 + q_2^2 + q_3^2)^3 \\ g_3 = a p_3^3 (q_0 + b q_3)(q_0 + c q_3)(q_0 + i\,q_3)^5 (q_0 - i\,q_3)^5 \end{cases} \qquad (30)$$

where F_1, $F_2 \in \mathbb{C}[q_0, q_1, q_2, q_3]$. The polynomials g_1 and g_2 both have 910 terms while g_3 has 24. We test `PHC` in windows 7 command prompt using the compiled executable file `phc.exe` provided by its authors compared with our interpretive code `PolynomialFactor` in Matlab. To level the base of accuracy comparison, we disabled the Gauss-Newton iteration option in our `PolynomialFactor` in this test since the iterative refinement was not developed for the computed factors when `PHC` was released. Table 2 lists the elapsed execution times and the errors, where the forward errors are measured on the known factors only. Since the three implementations are tested on different platforms, the comparisons should be considered indirect. Nonetheless, the results appears to show our algorithm is efficient and accurate on those polynomials. It also appears that `PHC` has been improved substantially as it runs much faster and outputs more accurate factors than it is reported in 2004. It also needs to be pointed out that `PHC` is based on homotopy method that is naturally parallel and is generally much faster on parallel computers. The test is conducted on sequential machine since, at this point, parallelization of `PolynomialFactor` has not been considered.   □

Example 5 (A polynomial with 5 numerical factorizations) An issue of significant importance on the concept of numerical factorization is that a polynomial may have

		g_1	g_2	g_3
Maple `factor`		not designed for empirical data		
`appfac`		$-\ -\ -\ -\ -\ -\ -\ -\ -$		
PHC	elapsed time	1382.8	1410.1	1.48
	backward error	9.6×10^{-10}	0.9945	1.7×10^{-12}
	forward error	1.3×10^{-12}	1.1×10^{-12}	3.4×10^{-13}
PolynomialFactor	**elapsed time**	**376.5**	**480.3**	**0.79**
	backward error	$\mathbf{4.3 \times 10^{-15}}$	$\mathbf{4.1 \times 10^{-15}}$	$\mathbf{4.8 \times 10^{-14}}$
(without refinement)	**forward error**	$\mathbf{2.0 \times 10^{-15}}$	$\mathbf{8.5 \times 10^{-16}}$	$\mathbf{5.0 \times 10^{-16}}$

Table 2 Factorization results on polynomials in (30) derived from Stewart-Gough platforms.

different numerical factorizations within different error tolerances approximating different conventional factorizations. A numerical factorization algorithm in this context needs mechanisms for targeting specific factorizations. For example, the polynomial

$$
\begin{aligned}
p_1 \;=\; & x^7 y + x^5 y^3 + x^6 - x\,y^7 - x^3 y^5 - 3\,x^2\,y^4 + 2(x^3\,y^3 - x^2\,y^6 + x^6\,y^2 - x\,y^5 - y^6 + x^4) \\
& + 6\,x^2 y^2 + 7\,x^4 y^2 + 4\,x^5 y + .999001\,x\,y^3 + 1.998002001\,x\,y + 4.999001\,x^3\,y + 2.999001\,y^2 \\
& - 1.000999\,x^2 + .001(y^3 + x^3 + 3\,x\,y^2 + x^3 y^2 + 3\,x^2 y + x^4\,y + x\,y^4 + x^2 y^3 + 2\,y + 2\,x) \\
& - 2.001997998999
\end{aligned}
\tag{31}
$$

can be considered as empirical data of p_1 itself and any one of the four factorable polynomials

$$
\begin{aligned}
p_2 &= (xy+1)\,(.001(x^2 y + xy^2 + y^3 + x^3 + 2x) + 6x^3 y + 2x^5 y + 2x^2 y^2 - 1.000999 x^2 + 2x^4 \\
&\quad + x^6 - y^6 + x^4 y^2 + 4xy - x^2 y^4 + 2.999001 y^2 - 2xy^5 - 2.001997999 + .002x - 2xy^3) \\
p_3 &= (-2xy^3 + 2xy + 2x^3 y + x^4 + 2y^2 - y^4 - 1.000999 + .001y + .001x)(x^2 + y^2 + 2)(xy+1) \\
p_4 &= (x^3 - xy^2 + x + x^2 y - y^3 + y + x^2 - y^2 + 1.001)(x+y-1)(x^2 + y^2 + 2)(xy+1) \\
p_5 &= (x+y+1)(x^2 - y^2 + 1)(x+y-1)(x^2 + y^2 + 2)(xy+1)
\end{aligned}
\tag{32}
$$

with data errors of various magnitudes listed in Table 3. In other words, the polynomial p_1 has a numerical factorization $1 \cdot p_1$ within an error tolerance between 0 and 10^{-14}, and numerical factorizations within error tolerances roughly in the intervals $(10^{-13}, 10^{-12})$, $(10^{-9}, 10^{-8})$, $(10^{-6}, 10^{-5})$ and $(10^{-3}, 10^{-2})$ approximating the exact factorizations of p_2, p_3, p_4, p_5 in (32) respectively. Our formulation of the numerical factorization includes the error tolerance ϵ and our implementation `PolynomialFactor` provides such an option. Based on the choices of those error tolerances, `PolynomialFactor` calculates all five numerical factorizations (32) with forward accuracies in the same orders of the data errors as shown in Table 3. Other algorithms such as Maple `factor`, PHC and `appfac` are designed to compute one numerical factorization from a given polynomial data.   □

10 Conclusions

Conventional factorization is an ill-posed problem in the sense that it is infinitely sensitive to data perturbations. The reason for such hypersensitivity is revealed by the geometry of polynomial factorization and the singularity can be quantified by the dimension deficit of the factorization manifold. The numerical factorization

underlying polynom-ial to be factored	p_1	p_2	p_3	p_4	p_5
data error $\sin(p_1, p_j)$	0	7.4×10^{-14}	1.1×10^{-10}	2.6×10^{-7}	4.9×10^{-4}
Maple `factor`	0	$----$	$----$	$----$	$----$
`appfac`+refinement	$----$	$----$	$----$	2.4×10^{-7}	$----$
PHC (no refinement)	$----$	$----$	1.4×10^{-9}	$----$	$----$
PolynomialFactor	$\mathbf{7.0 \times 10^{-16}}$	$\mathbf{8.9 \times 10^{-14}}$	$\mathbf{9.2 \times 10^{-11}}$	$\mathbf{2.1 \times 10^{-7}}$	$\mathbf{5.6 \times 10^{-4}}$

Table 3 Forward accuracies of Maple `factor`, PHC, `appfac` and `PolynomialFactor` from the data polynomial p_1 in (31) calculating the factorization of either p_1, p_2, p_3, p_4 or p_5 (32).

as formulated in this paper generalizes the concept of conventional factorization and eliminates the ill-posedness. By establishing the fundamental theorems for geometric structure of multivariate factorization, we proved that the numerical factorization uniquely exists and possesses Lipschitz continuity with respect to data under the overall assumption that the data error is small. Consequently, the numerical factorization achieves the objective of recovering the factorization accurately even if the polynomial data are empirical and the accuracy is in the order of data precision. An algorithm is implemented as a Matlab module and numerical results support this conclusion.

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